Tarski’s Thesis

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"Tarski’s Thesis" is the claim that a certain invariance condition can serve as our criterion of logicality. My goal in this chapter is to explain the thesis, provide it with a philosophical justificiation, and respond to three recent criticisms due to Solomon Feferman.

CRITERION OF LOGICALITY

In a 1966 lecture, "What are the Logical Notions?", Tarski proposed the following criterion of logicality:

Invariance under Permutation: A notion is logical if it is invariant under all permutations of the individuals in the "world" (or universe of discourse).1

by "notions" Tarski understood not linguistic or conceptual entities but objects of the kind referred to by such entities, i.e., objects in the world, including individuals, properties (yes), relations, and functions. "World" he understood as including both physical and mathematical objects and as forming a type-theoretic hierarchy, based on Principia Mathematica as a similar theory. In the present context it will sometime be convenient to view objects as operators (characteristic functions representing them) and not standard set theory with urelements rather than Principia Mathematica as our background theory. By centering his attention on objects or operators (worldly entities) rather than concepts (linguistic entities) Tarski follows the (pseudonymous) truth-functional definition of logical connectives in propositional logic. This definition I must confess writing this chapter while visiting the philosophy department at the University of Santiago de Compostela in Spain. I would like to thank the participants in my philosophy of logic seminar, and in particular Óscar Marce, Juan Miguel Agustín, and Luis Viñas for stimulating conversations on issues related to this chapter. I am also thankful to the group in Barcelona and to the participants in the conference "Practical Issues in Logic: Logical Consequence and Logical Commitments Revisited" for their thoughts and comments. I am also thankful to Dean Bumby for his suggestions and written comments, and to Peter Poole for his comments.

1 Tarski (1956: 149).

identifies logical connectives with certain objects, namely, Boolean truth fun-
sions, and it is these objects, rather than the names or descriptions used to refer to
them, that are said to capture the idea of logicality on the propositional level. One
advantage of the objective route is that it avoids complications arising from the
varieties of linguistic usage.2 Another advantage is the existence of a richer, more
precise, and more sophisticated machinery for talking about operators than about
concepts.

Before examining Tarski’s specific criterion, let us consider the idea of a general
criterion of logicality independently of its content. What is the purpose of such a
criterion? What would a systematic principle that demarcates the logical from the
non-logical (not just on the level of propositional connectives but also on the level
of quantifiers and other non-propositional operators) accomplish? The answer, I believe,
is this: First, it would bring an end to the current practice of an ad hoc, utterly un-
informative, definition-by-enumeration of the logical operators other than connectives.
Second, it would solve a serious problem that threatens to undermine Tarski’s model-
theoretic definition of logical consequence, and with it the entire field of logical
semantics. Furthermore, such a principle would considerably deepen our understand-
ing of the nature of logic, expand our ability to approach logic critically, create a
terrible domain of mathematical investigations, help solve outstanding problems in lin-
guistic semantics, and perhaps make other contributions as well, e.g., explain the rela-
tionship between the concept of logicality and other central philosophical concepts,
explain logic’s relation to neighboring fields (both within and outside philosophy),
and so on.

One would have expected Tarski to motivate his criterion by the problem that
threatened his own definition of "logical consequence," and whose full import he
recognized and brought to our attention (Tarski 1956b, namely, the problem that
the definition’s adequacy depended on the existence of an adequate criterion of logic-
ity. At the time Tarski worried that such a criterion would never be found (in which
case his definition would be forever unjustified), and this naturally leads us to expect
that his 1966 lecture was intended to answer those worries.

However, judging from what Tarski explicitly said (and did not say) in his 1966
lecture, his route to the criterion of logicality was completely divorced from his early
concerns.3 Instead, Tarski arrived at this criterion based on general considerations
concerning the demarcation of fields of knowledge. His starting point was Klein’s
demarcation of geometrical fields based on their invariance properties. Klein sug-
gested that each geometrical field could be characterized by the invariance condition
satisfied by its notions. This condition had the form:

Geometric Invariance: Geometric notions are invariant under all 1-1 transfor-
mations of the geometrical space onto itself which preserve X.

2 See Sher (2003). A similar advantage accrues to the objective, model-theoretic definition of
logical consequence as opposed to the linguistic, substitutional definition of this concept. (See
Tarski 1936 and Hume 1994.)

3 One of the things that Tarski explicitly said (p. 149) is that he was not interested in the problem of
logical consequence (i.e., as he put it, logical truth) in that lecture.
By strengthening X we obtain the transformations we seek into account, getting more specific geometrical notions, by weakening X we are obtaining the transformations taken into account, getting more general notions. Thus, if X is the requirement that the ratio of distances between points be preserved, the class of notions satisfying Geometric Invariance is the class of Euclidean notions. By strengthening X we get the requirement that actual distances between points be preserved, we obtain a characterization of our present geometric notions, namely those applicable to rigid bodies which don't change their shape under movement (or transformations); and by weakening X to the requirement that If I express by noting that openness (openness) be preserved, we obtain a characterization of rigid geometric notions, namely the topological notions. Now, Tarski asked: What would happen if we weakened X too much to be possible, i.e., if we set no requirements on the transformations taken into account? Then, we would get the condition.

General Invariance: No notion O is invariant under all 1-1 transformations of space, or the universe of discourse, or the "world" onto itself for under all permutations of the "world".

This invariance condition has all 1-1 transformations into account and, as a result, characterizes all the notions. What is the character which studies these notions? Tarski suggested that this character is: the one that deals with our most general notions, notions which are invariant under all 1-1 transformations of the world onto itself.

Today, we usually allow a slightly different version of Tarski's criterion. In fact, Tarski's (1966) concept remained unknown for many years, and the current version is basically due to Leibniz (1966) generalization of Montague's (1957). The term "involves" "isomorphic" (or "bijective") instead of "permutations" (or "transformations") and refers to a total "structure" rather than to a total, universal, "world." One way to formulate this criterion is:

Invariant under Isomorphism: An operator O is logical if it is invariant under all isomorphisms of its argument structures where:

(i) A structure S is an m-tuple, m ≥ 1, whose first element is a unique, A (i.e., a non-empty set of objects treated as individual), and whose other elements (if any) are certain ordered constructs of elements of A.

(ii) Two structures, (A, β1, . . . , βn) and (A', β1', . . . , βn'), are isomorphic—(A, β1, . . . , βn) ≃ (A', β1', . . . , βn')—if A = A' and there is a bijection f from A to A' such that for every 1 ≤ i ≤ n, βi is the image of βi under f.

(iii) An operator O represents an object o of a given type—an individual, a property of individuals, an n-place relation of individuals (n ≥ 1), an n-place function from individuals to an individual, a property of properties of individuals, a monadic first-order quantifier, a relation of properties of individuals (i.e., a relational first-order quantifier), a property of relations of individuals (i.e., a polyadic quantifier), etc.—and specific n arguments (or co-terms) in each universe.

Specifically:

1. An operator representing an individual o assigns to each universe A a 0-place function whose fixed value is o ∈ A, and which is treated in some conventional manner reference.

2. An operator representing a first-order property assigns to each universe A a function from all members of A to a truth value (which, physically, we assume is T or F).

3. An operator representing an n-place function f (x1, . . . , xn) assigns to each universe A a function from all n-tuples of members of A to (T, F).

4. An operator representing a first-order monadic quantifier assigns to each universe A a function from all subsets of A to (T, F).

5. An operator representing a first-order binary relational quantifier assigns to each universe A a function from all pairs of subsets of A to (T, F).

6. An operator representing a first-order polyadic quantifier of the simplest type assigns to each universe A a function from all binary relations on A to (T, F).

Etc.

(iv) If O is an operator whose arguments are of types t1, . . . , tn, A is a unique, and \( β1, . . . , βn \) are constructs of elements of A of types t1, . . . , tn, respectively, then \( β1, . . . , βn \) are arguments of O in A (or \( β1, . . . , βn \) is an argument of O in A) and \( (A, β1, . . . , βn) \) is an argument structure of O.

For example:

(a) The first-order property "is red" is represented by an operator, R, which for every universe A assigns a function, RA : A → (T, F), such that for any a ∈ A, RA(a) = T iff a ∈ R. (The argument structures are structures (A, a), where A is a universe and a ∈ A.)

(b) The first-order identity relation is represented by an operator, =, which for every universe A assigns a function, RA : A × A → (T, F), such that for any a, b ∈ A, RA(a, b) = T iff a = b. (The argument structures are structures (A, a, b), where a, b ∈ A.)

(c) The first-order existential quantifier is represented by an operator, ∃, such that for every universe A assigns a function, RA : A → (T, F), and for every B ⊆ A, RA(B) = T iff B is not empty. (The argument structures are structures (A, B), where B ⊆ A.)

(d) The first-order quantifier "for all" is represented by an operator, ∀, such that for every universe A assigns a function, RA : A → (T, F), and for every B ⊆ A, RA(B) = T iff B is non-empty. (The argument structures are structures (A, B), where B ⊆ A.)

(e) The first-order monadic quantifier "is a property of a human" is represented by an operator H of the same kind as ∃, and such that for every B ⊆ A, H(B) = T iff B is the set of all individuals.
iff all the members of B are humans. (Its argument-structures are the same as those of $\mathfrak{S}$.)

The preceding polyadic quantifier "is a well-ordering" is represented by an operator $\mathfrak{W}$ such that $\mathfrak{W}A, PA \to (T, F)$, and for every $R \subseteq A \times A$:

$$\mathfrak{W}_R = T \iff R \text{ is well-ordered}. (A, R)$$

And so on.

We now define:

An $n$-place operator $\mathfrak{O}$ is invariant under all isomorphisms in its argument-structures if

for any of its argument-structures $(A, \beta_1, \ldots, \beta_n)$ and $(A', \beta'_1, \ldots, \beta'_n)$, iff $(A, \beta_1, \ldots, \beta_n)$, in $(A', \beta'_1, \ldots, \beta'_n)$, then $\mathfrak{O}_A(\beta_1, \ldots, \beta_n) = \mathfrak{O}_{A'}(\beta'_1, \ldots, \beta'_n)$.

It is easy to see that all the standard logical operators—e.g., (b) and (c), as well as the logical connectives when considered as relational operators—are logical according to this criterion, and that all trivially non-logical operators—operators like (a) and (d)—are not. But the Invariance-under-Isomorphism criterion is a substantive criterion that does not just repeat what we think of as logical to a systematic, theoretical judgment. Quantifiers like the infinitary (a) and (d) are also logical. Other non-standard logical operators include the uncountability quantifier and the monadic and relational "most." In general, mathematical operators as they appear in first-order theories—e.g., the first-order set-membership operator (c)—are not logical, but when raised to a higher order—e.g., the second-order set-membership operator (c)—they are logical.

* * *

4. e.g., the logical connective "$\land$" when considered as an n-ary operator (as in what it appears in an open formula of the form $B \land C \lor D$) is represented by an operator $\mathfrak{O}$ such that $\mathfrak{O}(PA, PB, PC, PD) = PA \land PB \land PC \land PD$ and for every $B, C, A, C, D, D \in C(\mathfrak{O} B, C, D) = T$ if the intersections of $B$ and $C$ (Its argument-structures are structures $(A, B, C, D)$ where $B, C, D \in A$.)

5. There is defined a follows:

(i) The first-order numerical quantifier "there are exactly a members" is represented by an operator $\mathfrak{U}_a$ of the same kind as $\beta$, and such that for every $B \subseteq A$:

$$\mathfrak{U}_0(\beta) = (T, F)$$

and $\mathfrak{U}_a(\beta)$ is represented by an operator $\mathfrak{N}_a$ of the same kind as $\beta$, and such that for every $B \subseteq A$:

$$\mathfrak{N}_0(a) = (T, F)$$

and $\mathfrak{N}_a(\beta)$ is represented by an operator $\mathfrak{F}^a_n$ such that $\mathfrak{F}^a_n(\beta) = (T, F)$, and for every $B C, A, a, B, C, D \in C(\mathfrak{F}^a_n B, C, D) = T$

6. The second-order membership relation is represented by an open operator $\mathfrak{S}$, of the same type as $\beta$, such that for any $B \subseteq A, a, \beta, Pa, \beta, P \mathfrak{S}$ is a set of $A$ and a is a member of $B$. (Its argument-structures are the same as those of $\beta$.)

7. For our purposes, we have to consider the set of n-positive integers; i.e., the set of the positive integers.

8. * "Feuer" here means "positive number."
Under transformations preserving openness are more general than those invariant under transformations preserving the type of distance, and the latter are more general than those invariant under transformations preserving actual distances.

To obtain the next general notions we require all restrictive conditions on the transformations guarding in the invariance condition. And invariance under all (finite-type) transformations characterizes the logical notion. The distinctive mark of logicality, on this conception, is that it arises generally, and this trait is captured by the Invariance-under-homomorphism (or permutation) criterion.

(That Tarski says)

Now suppose we continue this idea, and consider all wider classes of transformations. In the extreme case, we would conclude the class of all one-one transformations of the space we are

sense of distance, or field this sort. What will be the scope of the invariance notion which deals with the invariance invariance under this subset class of type-preserving? Here we will have very invariances, all of a type of general character. I suggest that they are the logical notions.

It is natural to associate certain generality with another characteristic feature of logic, the topic neutrality, and this seems to strengthen the plausibility of interpreting Invariance-under-homomorphism as maximal generality.

But does Invariance-under-homomorphism yield the most general notions? In "logic" and "invariance" (2006) Dennis Bostrov challenges the identification of Invariance-under-homomorphism with maximal generality.

The interpretation in terms of generally acts on the assumption that it is under the biggest class of transformations yields maximal generality. The idea is that the group of all transformations is to be a big one with, because in that case the transformations do not respect any extra-structure, such as, e.g., the topological structure of the space. Let us have a closer look at this idea. Permutation invariance is just one act that is in an automorphism linking (M, A) and (M, A*), a quantifier E acting as M has to give A and A* the same value. On the one hand, this is indeed liberal, because no further structures beyond the extension A and A* do enter into account. But on the other hand, this is quite demanding: for (M, A) and (M, A*), to be similar from a logical point of view, they have to share exactly the same structures. That is, they have to be isomorphic. Now there are a lot of other concepts of similarity between structures which are used in both theories and to which we are less demanding. Instead of requiring the structure to be fully general, one lowers the requirement to some kind of approximate preservation. Why should we require that every one of them generally in a good way to approach logicality, there is no evidence that the class of all permutations is the best applicable for the job.

Bostrov's point is well taken. In the sense case we can remove all constraints on the functions involved, requiring logical operators to be invariant under all functions (from argument-structures to argument-structures of a given kind) whatever. This would give us the natural generality notions (in one reasonable sense of the word), but these notions would have very little to do with what we think of as logic. All the standard logical notions would fail this criterion, and the notions that would satisfy it would be such notions as: "is an individual," "is a property of individuals," "is an in-place relation of individuals (n ≥ 1)." "is a property of properties of individuals," etc. Logic, according to this characterization, would be a theory of semantic space, not a theory of inference (or manipulation of proofs) as we intend it in the first. I conclude that: (a) Invariance-under-homomorphism does not mean structural generality, and (b) if we want to preserve any semblance to what we intuitively mean by logic, we cannot regard structural generality as for that matter topic neutrality, or the mark of logic.

Formality

On my interpretation (Bostrov 20011) and elsewhere, the invariance-under-homomorphism criterion in a criterion of formality or structural invariant structures or the formally identical—identity-up-to-homomorphism is formal identity. The basic idea is that logic is a theory of reasoning based on formal (structural) laws governing our thinking on the one hand and reality on the other, and the Invariance-under-homomorphism criterion says that to be formal is to be isomorphic structures as the same structures.

Formal operators do not distinguish between isomorphic structures (or rather between isomorphic argument-structures), but some formal features of arguments are on the formal status of the underlying argument.

The view that Invariance-under-homomorphism captures the concept of formality (or structurality) is well-known from the philosophy of mathematics. Structuralism, in particular, view mathematics as the science of structure (or formal structures), and Invariance-under-homomorphism as a mark of structuralism. The Invariance-under-homomorphism criterion characterizes logic as a theory of formal or structural inference, based on the laws governing formal or structural operators.

What is the relation between logic and mathematics under this interpretation? I will attempt to answer this question in the next section, but in the meantime let me say that on the "formalism" conception of logic, logic and mathematics are interrelated theories, approaching the same topic, the formal, from different, yet intertwined, perspectives. Mathematics investigates the laws of formal structure; logic applies these laws in general reasoning. Logic includes mathematics, rather than being a higher-order, so it can be applied in inference in general. The idea is that formal operators—union, intersection, complementation, non-emptyness, majority ("more"), finiteness, and others—are applicable to structures of objects studied in all areas of knowledge, and therefore, informal inferences based on the laws governing them are valid in all areas.

This universal applicability of the formal operators explains logic's generality and topic neutrality. Logic is not distinguished between different topics of discourse since the formal laws governing the behavior of individuals, properties, and relations in different areas are the same. On all areas individuals are identical to themselves, the union of non-empty properties is non-empty, etc. Their differences concern something other than those formal laws, and logic abstracts from such differences. Comparing the two characterizations of logic associated with the Invariance-under-homomorphism criterion, then, we can say that the formality of logic, measured in generality (not structural generality, but a very high degree of generality), while the generality of logic does not mean its formality. This is but one advantage of taking formality
than generality as the mark of logic. In the remainder of this chapter I will examine

PHILOSOPHICAL JUSTIFICATION OF THE INVARIANCE-UNDER-ISOMORPHISM CRITERION

Now that we have a basic understanding of the invariance-under-isomorphism criterion, the next task is to provide a philosophical justification for it. I think it is now clear that this criterion satisfies the four methodological desiderata mentioned above, namely, systematization and informativeness (i.e., a genuine principle of logic as opposed to a definition by enumeration). But it also violates the other desiderata. For example, it has opened new areas of research in mathematics and linguistics and helped solve standing problems in both disciplines. Here, however, I would like to focus on substantive philosophical points that support this criterion, i.e., give it what may be called "a foundational justification." By this I mean showing how the philosophical conception of logic associated with this criterion—namely, the "formal" conception briefly discussed in the last section—is capable of providing a foundation for logic largely due to its association with this criterion.

Methodological quarrel: holistic vs. foundationalist foundation

In thinking about a foundation for logic most of us think in foundationalist terms: we think that the only way to establish logic is by using epistemic resources that are more basic than those produced by logic itself. And this leads us to a presumptuous conclusion: since no sufficiently rich branch of knowledge is more basic than logic, there is no possibility of establishing logic a foundation for, or a justification of, logic. Such a conception of logic is in principle impossible. The source of the problem, it is easy to see, is the foundationalist conception of the foundation (justification, grounding) relation as intuitively strongly ordered. In the ideal case, foundationalism requires that our entire system of knowledge be ordered by an absolute partial-ordering, that is, according to a ranking of minimal (initial, atomic) elements, and that all non-minimal elements

14 Among other things, the invariance-under-isomorphism criterion does not obstruct logical analysis where the Buckingham-Franks functional criterion did for propositional logical operators, namely, provide a complete, precise, axiomatic, definitive, blocking out other meanings, and explaining how they work, a "concrete" or "better" definition of logical operators. See a definition formalized in Sher (1979), Ch. 6, and a (sectionally) described in Sher (1996b).

15 For example, if a list of the development of "model-theoretic logic," and "grounded quantifier theory" (see remarks made on their new fields in the introduction to this volume, Klavins and others in this issue) indicates that we would have done logic, with the question of universality, to the problem of demarcation in linguistic semantics, and the question of partial functions and branching quantifiers in natural language. These two are basic to the three. For a small sampling see Kripke (1979), Lakatos (1974), Barwise and Feferman (1983), Hig-


in the system be conceptual as each minimal element grounding it by a finite chain. This central feature of the foundationalist method is so Achilles heel due to it, foundationalism has, in principle, no resources for grounding the basic constituents of knowledge—the disciplines constituting the basic sciences in the foundationalist hierarchy. In particular, foundationalism is incapable of providing a foundation for logic. As a basic branch of knowledge, logic can participate in the foundation of other sciences, but no science (or combination of sciences) can provide a foundation for logic. Having posited (i) that any resource for grounding logic must be more basic than the resources produced by logic itself, and (ii) that there are no (or not enough) resources more basic than those produced by logic, foundationalism is incapable of grounding logic.

In view of these considerations, it is clear that a foundation for logic must be holistic. I will not be able to explain in great detail the idea of a holistic foundation, or foundationalism, here. (For an extended discussion see Sher (2006).) But a few points have to be made:

- Foundational holism would provide a foundation for logic in the sense of describing its basic mechanisms, justifying the definitions of central meta-logical concepts, solving standing problems in the philosophy of logic, identifying conceptions on logic, elucidating the relation between logic and related concepts, sorting out and accounting for the distinctive characteristics of logic, explaining logic's role in our systems of knowledge, throwing light on the relation between logic and mathematics, providing critical tools for detecting errors and making improvements in logical theory, etc.

- Foundationalism is holistic in reality. Its opposition is foundationalism. By requiring that every branch of knowledge be grounded in reality, the holistic, it permits us to use all resources available to us in providing such a grounding.

- Foundationalism does not require an absolute, infallible foundation but it requires a solid foundation.

- Foundational holism requires a theoretical, and that just intuitive, grounding of logic.

- Foundational holism rejects vicious circularity, but not circularity per se. Which circularity is vicious is determined by holistic methods.

Having made these methodological points, I will proceed to show how the formalist conception of logic as exemplifies part of the foundationalist task mentioned above due to its association with the invariance-under-isomorphism criterion.

A. Explanation of logic's constitution as truth is a commonplace to say that as a theory of logical truth (truth of a certain kind) and of logical consequence (transmission of truth of a certain kind) logic is intimately connected with truth. But what, exactly, is the nature of this connection and its constitution on logic? Let us start with general theoretical considerations.

Assuming the classical idea that truth importantly involves some correspondence relation between truth-bearers and reality, let us consider two truth-bearers, y, and
S₁, whose truth-conditions straightforwardly and paradigmatically exemplify this idea. For the sake of simplicity, let us further assume that S₂ and S₃ are distinct and non-contradictory. Now, suppose that according to some logical theory, L, S₁ is a logical consequence of S₂. In symbols:

(1) (Level of Logic) \( S₁ \vdash_{L} S₂ \)

Further, suppose that S₄ is true. Then (1) says that the truth of S₁ extends to, or is transmitted to, or is preserved by, S₄:

(2) (Level of Language) T(S₁) → T(S₄).

(1) says something weaker than that, but let us attend to the weaker claim first.

Let \( \mathfrak{D}_1 \) and \( \mathfrak{D}_2 \) be the situations that have to be realized for S₁ and S₄ to be true and that would guarantee their truth were they to be realized. Figuratively:

(3) (Level of Language) T(S₁) \( \mathfrak{D}_1 \)

(Level of World) \( \mathfrak{D}_1 \) \( \mathfrak{D}_2 \)

Now, suppose that in the world \( \mathfrak{D}_1 \) is the case but \( \mathfrak{D}_2 \) is not. (In the extreme case, \( \mathfrak{D}_1 \) makes \( \mathfrak{D}_2 \) not true.) Let \( \mathfrak{D}_1 \) and \( \mathfrak{D}_2 \) be the cases that have to be realized for S₁ and S₄ to be true and that would guarantee their truth were they to be realized. Figuratively:

(4) (Level of Logic) S₁ \( \vdash_{L} S₄ \)

Logic, indeed, is constrained to reach more deeply than the above considerations suggest. Suppose that in the world-true \( \mathfrak{D}_1 \) and \( \mathfrak{D}_2 \) are the case. Not \( \mathfrak{D}_2 \), being the case, does not require \( \mathfrak{D}_1 \) being the case:

(5) (Level of World) (\( \mathfrak{D}_1, \mathfrak{D}_2 \)) \( \mathfrak{D}_1 \Rightarrow \mathfrak{D}_2 \)

Thus, again:

(6) (Level of Language) S₁ \( \Rightarrow_{L} S₄ \)

Now, an adequate criterion of logicality has to explain, to be incorporated in an account that explicates, this a priori constraint on logicality. The formal interpretation of the invariance-under-bisimulation criterion delineated in the last section is embodied in a "formal" account of logic that does just that. We can sum up its main points as follows:12

(i) The logical constants of truth-bearers—especially, their logical constants—represent formal properties, relations, and functions, where formality is interpreted as invariance-under-bisimulation.

12 "\( \Rightarrow_{L} \)" is a symbol of an unspecified kind for logical consequence \( S₁ \Rightarrow_{L} S₂ \) read: "S₁ logically implies S₂."

For a more detailed account see Svet (1991) and related papers.

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(iii) The logical form of truth-bearers is obtained by holding their logical constants fixed and viewing their non-logical constants as variable.

(iv) Corresponding to a truth-bearer S is a situation, \( \mathfrak{D} \), that would make S true if it were to be realized. Corresponding to the logical form of S is the formal skeleton of \( \mathfrak{D} \), which contains only parameters of \( \mathfrak{D} \) which correspond to its logical constituents. For example, corresponding to “Something is white and round” is a structure, \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \) where \( \mathfrak{A} \) is the intended universe of discourse, B is the collection of white things in \( \mathfrak{A} \), C is the collection of round things in \( \mathfrak{A} \), and the intersection of B and C is not empty. The formal skeleton of \( \mathfrak{D} \) contains the formal parameters of \( \mathfrak{D} \) corresponding to the logical constants of S, namely: intersection and non-emptiness (a cardinality parameter).

(v) Logical consequence is a relation between truth-bearers which represents a universal formal law connecting the situation corresponding to the “premise” truth-bearers to the one corresponding to the “conclusion” truth-bearer. Alternately, logical consequence corresponds to, and is largely due to, a law connecting the formal skeleton of the “premise” situations to the formal skeleton of the “conclusion” situation. This law is universal in the sense that it holds in all formally possible situations, or in all possible formal structures. For example:

Something is white
\( \Rightarrow_{L} \) a logical-consequence of

Something is white and round

because it is a formal law that whenever an intersection of two subsumes is not empty, the first of these subsumes is not empty; it is not a logical consequence of

Something is white or round

because it is not a formal law that whenever a union of two subsumes is not empty, the first of these subsumes is not empty.

If we regard formal laws as formally necessary, we can concisely represent the present conception of logical consequence, thus:

(Level of Language) S₁ \( \Rightarrow_{L} S₄ \)

(Level of World) \( \mathfrak{D}_1 \) formally necessitates \( \mathfrak{D}_2 \).

In contemporary (Tarlockian) semantics we represent the formally possible situations via a given language by the totality of models for that language. Universal formal laws are represented by regularities across all models.

This explains the standard (Tarlockian) semantic definition of logical consequence S is a logical- consequence of K iff S is true in all models i.e., formally possible situations in which all the members of K are true, i.e., when S is a logical consequence of K, this is due to some formal law, connecting the situations corresponding to S and K.
supported by it. According to this explanation logic plays a dual role in knowledge: it sets general constraints on what counts as knowledge, and in ways it creates useful tools for expanding and constructing our knowledge. Let us consider the latter first.

Expansion of knowledge

Being finite and relatively short-living creatures, we cannot hope to establish all our knowledge directly, but have to resort to such indirect means as inference to obtain a considerable portion of our knowledge. In inference we use our knowledge of the relations between objects or situations plus some knowledge of these objects or situations to obtain new knowledge which, as inferred knowledge, does not require independent verification. For example, if we have knowledge about the chemical composition of objects and the relations between chemical structures, we can use this knowledge to obtain new knowledge about objects. But while chemical laws enable us to expand our knowledge in a small number of areas, formal laws enable us to expand it in all areas. Given that formal features of objects are consistent with the general laws, we can apply them to all areas of knowledge. This is how we gain knowledge of the general laws of objects, and how we can use them to expand our knowledge of objects. In this way, we can expand our knowledge of objects in a way that is consistent with our knowledge of the general laws of objects.

Evolution of knowledge

Given the prevalence of formal features of objects and our constant reference to such features in discourse and theorizing, the threat of formal errors in our systems of knowledge becomes large. But the logic of the general laws does not distinguish between different domains of knowledge, it is possible to take care of such errors in "one hit swoop", so to speak, i.e. it is a threat that persists in all (or most) fields of knowledge at once. This opportunity is seized by logic. Logic builds into our language principles that prevent us from making erroneous inferences in the general laws. For example, by telling us that conclusions of the form "\( \forall x \Phi(x) \land \forall x \Psi(x) \) are false (or that a combination of incommensurability of the form "\( \forall x \Phi(x) \) and "\( \forall x \neg \Phi(x) \) in consistent) logic prevents us from making certain errors concerning the behavior of objects under the completion of an operation (in any field). By telling us that inferences of the form "\( \forall y (\forall y \Phi(y) \land \forall y \Psi(y)) \) are invalid, it prevents us from mis-using certain symmetrically existing where they do not. And so on.
Gile Sher

Solution to Tarski’s Problem

In his 1936 chapter, “On the Concept of Logical Consequence,” Tarski sought a definition of “logical consequence” that would satisfy two intuitive criterions:

1. Certain considerations of an inductive nature will stem from our starting point. Consider our class $K$ of sentences and a sentence $X$ which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class $K$ contains only true sentences and the sentence $X$ is false. Moreover, since we are concerned here with the concept of logical, i.e., formal, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge and it is peculiar by knowledge of the object to which the sentence $X$ or the sentences of the class $K$ refer. The consequence relation cannot be affected by replacing the designation of the object referred to in these sentences by the designations of any other object. The two considerations just indicated... seem to be very characteristic and essential for the proper concept of consequence.


Based on these considerations Tarski formulated his “semantic definition of logical consequence”:

The sentence $X$ follows logically from the sentences of the class $K$ if and only if every model of the class $K$ is also a model of the sentence $X$.

(ibid.: 417)

Is this an adequate definition? Does it satisfy the intuitive criterions? At first Tarski gave a positive answer:

It seems to me that everyone who understands the content of the above definition must admit that it gives quite well with common usage. This becomes still clearer from its various consequences. In particular, it can be proved, on the basis of the definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences.

(ibid.)

But must be qualified his answer:

I am not at all of the opinion that in the result of the above discussion the problem of a materially adequate definition of the concept of consequence has been completely solved. On the contrary, I still see several open questions, one of which—perhaps the most important—I shall point out here.

(ibid.: 418)

This question was the demarcation of logical constants:

Underlying our whole conception is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not exact. Arbitrary. If, for example, we tried to include among the extra-logical such implication signs, or the universal quantifier, then our division of the concept of consequence would lead us in tangles which obviously contradict ordinary usage. On the other hand, no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be impossible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which seem to be sharp contrast to ordinary usage.

(ibid.: 418–19)

These qualifications led Tarski to conclude his chapter on a skeptical note:

Further research will most likely clarify the path which remains open. Perhaps it will be possible to find two separate objective arguments which will enable us to justify the traditional distinction between logical and extra-logical expression. But I also consider it to be quite possible that investigations will bring to the surface results in the direction, so that we shall be compelled to regard such expressions as logical consequence and... ontology as relative concepts which mean, on each occasion, to be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra-logical.

(ibid.: 420)

The Invariance-under-Isomorphism criterion offers a positive solution to Tarski’s problem. It offers a demarcation of logical constants under which Tarski’s definition of logical consequence can be shown to satisfy the intuitive criterions. To see how, consider the following:

1. Tarski set two intuitive criterions on an adequate definition of logical consequence:

C1) Necessity: When $X$ follows logically from $K$, $X$ follows necessarily from $K$.

C2) Formality: When $X$ follows logically from $K$, $X$ follows formally from $K$.

2. Regardless of what Tarski himself understood by necessity, if we show that his definition satisfies a robust standard of necessity, we will have shown that it satisfies whatever weaker standard he might have had in mind.

3. Formality can be interpreted both syntactically and semantically. Philosophers often think of formality as syntactically, but the key to vindicating Tarski’s definition is to think of it semantically.

4. Tarski himself offers the key to a semantic interpretation of formality:

(As a formal relation, logical consequence cannot be influenced in any way by knowledge of the objects to which the sentence $X$ or the sentences of the class $K$ refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects.

(ibid.: 414–15, cited above)

5. This paragraph suggests that the formal is characterized by its incapability to distinguish the identity of objects in a given universe of discourse. This is an invariant characterization: formal relations are invariant under replacement of objects. Now, if we interpret “replacement” as “1–1 and some transformation of mapping and “replacement of objects” as “replacemnt of objects of all types induced by replacement of the individuals in a given universe of discourse,” then we get the invariance-under-isomorphism criterion of logicality.
6. Under this criterion all logical operators are derivable from mathematical operators by raising them to a higher order (as we have done on p. 307), and in this sense they are essentially meta-mathematical.

7. But the laws governing mathematical operators are absolutely formal and necessary (where this necessity is at a relatively strong kind of necessity, stronger than biological, physical, and even metalevel metanecessity). Therefore, if logic's concept is due to the formal (or meta-mathematical) laws governing the logical operators, logical-conceptual is formal and logically necessary.

8. Now, on a formalistic reading of Tarski's definition does not simply the antecedent of this conditional. The totality of models represents the truth of formal possibilities; logical-conceptual preserves truth across all models; they do so due to the logical structure of the sentences involved; this logical structure reflects the formal skeleton of the situations described by those sentences therefore the preservation of truth is due to connections that hold between the formal skeletons of the situations involved with formal possibilities and formal connections preserving through the totality of formal possibilities are laws of formal structure. It follows that consequence satisfying Tarski's definition are formal and necessary as required by the intuitions constraints (however wrong the necessity constraints is when it is).

9. Explanation of the distinctive characteristics of logic

Logic is often characterized by its basicness, generality, topic-neutrality, necessity, formality, strong normative force, certainty, a-priority, and analyticity. While, as foundational beliefs, we notice the particular analyticity of logic and quality in a priority, we can explain its other characteristics (including qua-genre) based on the invariance-under-isomorphism criterion, i.e., explain why the laws of logic and its c-sequences are as basic, general, topic-neutral, formal, strongly normative, and highly certain as they appear to us to be, and to what degree they is a-priori.

We have already seen how the invariance-under-isomorphism criterion, either alone or together with other elements of the formalistic accounts, explains the formality.

10. In defending the adequacy of Tarski's definition it may seem that we have to confront Echternach's (1958) challenge to it, but it is false. Echternach considers two conceptions of logic the so-called representational and interpretational conceptions. But the formalistic conception of logic offered here in and in 1954 falls under interpretational. Since Echternach's (1958) conception of logic is a logical this makes no sense as facts. This includes his claim that the problem of logical constants is a "not having." Echternach regards the problem of logical constants is a "not having" because for strict logical constants do not pose a genuine problem in Tarski's definition, but because he claims that Tarski's definition is plagued by other problems as well and equally solving the logical constants problem will not by itself suffice as adequate. However, the additional problems Echternach allies it specific to the nonrepresentational element of logic and is not an issue on the formalistic conception. Therefore, on the contrary the problem of logical constants for being a "not having." — the main obstacle to the adequacy of Tarski's definition. For a fuller critique of Echternach's (1958) see Sher (1996a).

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generality, topic-neutrality, and necessity of logic. Let us, then, turn to the other characteristics.

Rationality and strong representational force. Logic is intuitively more basic than other disciplines. The grounding of geography, biology, and chemistry involves establishing their logical consistency, i.e., establishing that there logic obey the laws of logic, but the grounding of logic does not involve establishing that laws obey the laws of geography, biology, and chemistry. This gap is related to a gap in the representational force of logic and other disciplines. Chemistry, biology, and geography have to assert the necessity of logic, but logic need not assert to these structures. Logic has normative authority over these disciplines, but not vice-versa. The invariance-under-isomorphism criterion explains why this is so. Since chemical properties are not preserved under isomorphism, logic has a stronger normative property than chemistry. As a result, logic does not distinguish chemical differences between objects is and is subject to the laws governing chemical properties. But chemistry does distinguish formal differences between objects for example, it distinguishes between air atoms and gas atoms. So chemistry is subject to the laws of formal structure. For example, chemistry is bound by the law

\[ 2(x) \cdot \psi \rightarrow \neg \exists(z) \cdot \\
\]
Explanation of the relation between logic and mathematics

Ever since Frege, logic and mathematics have been treated as closely related disciplines whose theories require an explanation. And one of the least noted, but methodologically most important achievements, of Frege's logicism was the enormous economy it brought to philosophical tasks of explaining the nature of logic and mathematics and providing them with a foundation. By reducing mathematics to logic, logicism reduced two theories to one. Instead of having to explain both the nature of logic and the nature of mathematics we now had to explain only the nature of logic and instead of the monumental task of constructing a foundation for logic and a foundation for mathematics, we had the more manageable task of constructing a foundation for logic. However, the search for a foundation for logic (independently of mathematics) led nowhere. The most influential attempts to construct an account of logic that would complement logicism — Carnap's constructivism — has by and large been discarded, and this, together with the almost unanimous rejection of logicism in itself has left us, once again, with the extremely difficult task of providing an explanatory account and a foundation both for logic and for mathematics.

The formal account of logic, with its Invariance-under-Bottomupness criterion of logicality, offers an explanation of the relation between logic and mathematics that has the same methodological advantage as Frege's explanation without having its shortcomings. Like Frege's account, it reduces the two fields to one, hence the two foundational tasks to one. But this time it is logic that is reduced to mathematics rather than mathematics to logic. Or, alternatively, both logic and mathematics are reduced to the formal, mathematics, in this account, builds a theory of formal structure, and logic provides a method of inference based on this theory. I will call this the new approach "mathematization." If logicism is the view that mathematics is a logical foundation, mathematics is the view that logic has a mathematical foundation. But there is a considerable methodological advantage to mathematization over logicism. While today we no longer use foundation accounts of logic, not centered on mathematics, we do have a number of promising foundational accounts of mathematics not centered on logic, for example, the Platonist account, the naturalist account, and the structuralist account. It is true that three accounts assume logic in the background, but once mathematicians seek to give a bottom foundation one logic does not pose a special difficulty. Logic does not stand at the center of any of these accounts, therefore the centrality involved is (at least prima facie) not serious.

But the current situation is even more felt. Not only are several accounts of mathematics incompatible with logical formalism, one of these accounts, the naturalist account, is very close to it in spirit. This is reflected in the fact that mathematical structuralism and logical formalism share the same identity criterion: coordination under isomorphism. Invariance under isomorphism is the identity criterion of logical operators according to logical formalism, and it is also the identity criterion, or at least an identity-criterion of choice, of mathematical structures according to mathematical structuralism. Thus, Suppes says:

No matter how it is to be articulated, structurism depends on a notion of two systems that thoughtfully be "same" systems. That is quite... We... need to specify a relation among systems that amounts to "have the same structure". There are several relations that will do for this... The first is isomorphism, a relation (and respectable) mathematical notion... Informally, it is sometimes said that (isomorphic) "possesses structure".

(Shapiro, 1997: 90-1; my underline)

A purported implicit identity-criterion at one exists if one is isomorphic to any one of two systems is an isomorphism type... any two models of it are isomorphic to each other.

(Bell, 1985; my underline)

Indeed, it would be just as appropriate to call our account of logic, "logical structuralism" as to call it "logical formalism" (and to call the structuralist account of mathematics, "mathematical formalism" as to call it "mathematical structuralism").

Furthermore, we can achieve the same methodological goal without reducing either discipline to the other, namely, by tracing both mathematics and logic to the same entity, i.e., the formal (structure). Analytically, logic and mathematics develop in tandem from a basic engagement with the formal (the structural). We can represent this joint development using something like the following time line: In stage 1, we develop a rudimentary logic—mathematics which studies some very basic formal operators, say complementation, union, intersection, and inclusion. Based on this knowledge we develop, in stage 2, a logical framework for theories in general, and using it we develop a more sophisticated mathematical theory of formal structure (say, naive set theory). Based as this theory we develop, in stage 3, a more abstracted logical framework, say the logical framework of standard first-order logic with its standard logical operators (\(\land, \lor, \Rightarrow\)), and the truth-functional connectives. And using this framework we develop, in stage 4, a more advanced mathematical theory of formal structure (say, orthodox set theory). In stage 5 we use this advanced theory to develop a criterion of logical identity (for example, the Invariance, under isomorphism—variance) and a semantic definition of logical consequence (for

[My remarks about "structuralism" need not be seen as strictly referring to structuralism as a term is used in structuralism. In structuralism, one speaks of "structures" as something like physical objects that are observed and studied. Here I am using the term in a different way. I use it to refer to the idea that mathematical structures are interconnected and that we can speak of them as having the same structure. This is a different usage from the structuralism of physical systems.]
example, Tarski's model-theoretic definition), and based on these, an expanded logical framework—so-called generalized first-order logic (or standard second-order logic). And this process may continue using this enriched logic we may arrive at a still more powerful mathematics and, hence on, it is a stronger logic. And so on.

To deal with the formal in logic and in mathematics we operate on different levels. In mathematics we construct the formal as (for the most part) lower-order, in logic we construct it as (for the most part) higher-order. Take, for example, the notion of number or the notion of union, intersection, and complementation. In standard arithmetic numbers are individuals and the mathematical quantities are properties: in arithmetic we think of numbers, interments, and complementations are properties on individuals, but in logic they are properties on properties (or propositions). As studied in mathematics, these notions do not satisfy the condition under-mentioned condition, but as studied in logic, they do. And the same holds for other formal notions; for example, the membership relation of standard set theory is not logical, but the membership relation of higher-order logic is. (One more technical version of this account would say that the mathematics of some mathematical concepts as non-logical and others as logical. The members notion, for example, may, mean number as non-logical entities, but the background mathematical concepts he uses to talk, and formulate questions, about number—e.g., the concept of set membership—is logical.)

Tarski's task in the philosophical criticism of the new logical criterion for the relation between logic and mathematics is different from mine:

The question is often asked whether mathematics is a part of logic. Here we are interested in only one aspect of this problem—whether mathematical entities are logical entities, and vice versa, whether mathematical concepts are logical concepts, which is outside our domain of discussion. Since it is now well known that the whole of mathematics can be constructed within set theory, or the theory of classes, the problem reduces to the following one: Are there mathematical relations (or not)? Again, it seems clear that all real-world mathematical entities can be defined in terms of one, the notion of belonging, or the membership relation, the final form of our question is whether the membership relation is a logical one or the sense of my suggestion. The answer will seem disappointing. For we can develop set theory, the theory of the membership relation, in such a way that the answer to this question is affirmative; we can proceed in such a way that the answer is negative. So the answer is 'As you wish'.

(Tarski 1966: 151–2)

In my view, the new logical criterion leads to a more intricate and interesting answer to this question. It suggests that there is a division of labor between higher-order logic and mathematics, one that leads to different practices in the two disciplines. Logic and mathematics approach the formal from two different, though complementary, perspectives: one for both the similarities and their differences. Mathematics seeks to discover formal laws, logic seeks to implement them: mathematics is interested in the formal as it can be objective, logic is interested in the formal as it concerns thought or language. And our cognitive capacities are such that discovery is not systematized in terms of individuals and their properties, implementation—

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is a statement of properties and relations and especially in terms of properties and relations of properties and relations. The formal is differently presented in logic and in mathematics, but at bottom it is the same in both. For additional points and a slightly different perspective, see Shef (1991, chs 3 and 6).

Tools for justifying logic's claims and detecting its errors

By using mathematical tools as a basis for logic tools, we are licensed to use mathematics, and indirectly, the tools used to justify it and detect its errors as a tool for justifying and detecting errors in logic. For example, in the recent that mathematical or rational intuition is a tool for justifying mathematical assertions, it is also a tool for justifying the underlying logical assertions. Or to the extent that sometimes (if not early) physical discoveries have formal ramifications, they can be used to contribute or drive down as logical assertions. Or to the extent that a new, clean, or an old conjecture, is proved in mathematics, we may be to justify a logical rule of proof or a logical inference. For example, the newly discovered proof of Fermat's Last Theorem justifies all the homomorphologically fault of inference of the form:

$$(\exists x \phi(x) \wedge \exists y \psi(y)) \rightarrow \exists z (\phi(z) \wedge \psi(z))$$

and

$$(\exists x \phi(x) \vee \exists y \psi(y)) \rightarrow \exists z (\phi(z) \vee \psi(z))$$

for all $x > 2$ and $y$. So if we find compelling reasons for including the Continuum Hypothesis in our system of formal structural, we can use them to justify either the logical inference

$$(\exists x \phi(x) \wedge \exists y \psi(y))$$

or the logical inference

$$(\exists x \phi(x) \vee \exists y \psi(y))$$

and so on.

These are some of the foundational advantages of the Invariance-Under-Isomorphism criterion and the formalist theory of logic within which it is offered.

It should be noted that the Invariance-Under-Isomorphism criterion also contributes to a crucial step in the philosophy of logic. The prevalent philosophy of logic today adheres to the so-called 'first-order theory' which says that standard first-order logic is the whole of logic. Very few systematic or theoretical grounds have been advanced in support of this assertion, and for the most part it has been accepted without serious argument. The Invariance-Under-Isomorphism criterion challenges this thesis on several grounds. For one thing, it is a challenge to one of the few theoretical arguments used to support it; namely, the argument from completeness (Quine 1970). Investigations connected with this criterion (e.g., Kiefer 1970) have proved that standard first-order logic is definitely not the strongest (extensional) logic which

17 This epigraph is due to Frege (1975) who made similar points to those I have made to make.
has the virtue of being complete, namely first-order logic—for example, first-order logic with the added logical quantifier "there are uncountably many"—is already complete. More importantly, the invariance-under-homomorphism criterion demonstrates that a systematic, theoretical, philosophically anchored, highly explanatory, mathematically rich, and linguistically fruitful criterion of logicality is possible. In so doing it is a new, higher standard of justification for those concerning the scope of logic, standard that, as far as I can judge, has not been met by any of the known justifications of the first-order thesis.

Our final task before turning to Emanet's criticism is to show how the invariance-under-homomorphism criterion for objectual logical operators relates to the Boolean, truth-functional criteria for propositional logical operators (logical connectives).

Invariance-under-Homomorphism and Truth-Functionality. In making statements we usually work with two types of structures—objectual structures and propositional structures, and we use two types of operators—objectual operators and propositional operators. Thus, in making a statement of the form

\[ (\exists x)(B(x) \land \neg C(x)) \]

we first consider an objectual structure with two properties, B and C, then, working with the objectual operator \( \land \) (the objectual correlate of the propositional operator \( \land \), namely, conjunction), we focus our attention on B and the complement of C, next, working with the objectual operator \( \neg \) (the objectual correlate of the propositional operator \( \neg \), we shift our attention to the intersection of B and the complement of C, then, working with the objectual operator \( \land \), we consider the possibility that the intersection is not empty; and finally, thinking in propositional terms and using the propositional operator \( \neg \), we say that this possibility is not realized: nothing is both a B and a non-C.

Now, if we concurrently use operators of two types, objectual and propositional, each defined in terms of the corresponding structure, then we treat two things coordinately: criteria of logicality, each formulated in terms of the relevant structure. Invariance-under-homomorphism is a criterion of logicality for objectual operators, and Truth-Functionality is a criterion of logicality for propositional operators. How are they connected? The formalist answer is that the same idea—formality—lies in the bones of both criteria, and the same technical device—"invariance under isomorphism"—is used in both, but with respect to different structures:

(I) An objectual operator is logical if it is invariant under all isomorphisms of its argument structures, which are objectual.

(II) A propositional operator is logical if it is invariant under all isomorphisms of its argument-structures, which are propositional.

And we also need two related logical predicates, "satisfaction" and "truth"—the former applying to open formulas whose variables, if any, are all objectual, and the latter applying to closed formulas (between whose open operators (i.e., those added to the operators of their open subformulas) if any, are all propositional, and whose definitions accordingly are propositional operators.

What is an isomorphism of propositional structures? When two propositional structures are formally the same! Well, formality in the domain of propositions is, from the classical approach covariance adopted here, preservation of Boolean structure. And the Boolean features of propositional structures are a generalization of the Boolean features of objectual structures. So the basic parameter in this generalization is binary structure or complementarity, which is common to both objectual and propositional structures, and we can arrive from the objectual form of this parameter to its propositional form in three steps (this, in the simplest case, can be described as follows):

(i) Objectual step:

Given an object \( a \) in a universe A and a set of objects \( B \) proper to B in A, there are exactly two possibilities with respect to \( a \), exactly one of which is realized: \( a \) is a B, \( a \) is \( \neg B \) (complement of B in A), the latter being equivalent to \( \neg a \) not \( B \) (in A).

(ii) Situational step:

Given the situation \( s \) in which \( a \) is a B (in A), there are exactly two possibilities with respect to \( s \), exactly one of which is realized: \( s \) is not the case (non-being, the case being the complement of being-the-case).

(iii) Propositional step:

Given a proposition \( p \) corresponding to \( s \), there are exactly two possibilities with respect to \( p \), exactly one of which is realized: \( p \) is true, \( p \) is false (false being the complement of truth).

These steps connect objectual structures to propositional structures and form a bond between the logical criterion of objectual operators and the logical criterion of propositional operators: an operator, objective or propositional, is logical iff it does not distinguish the non-formal features of its argument-structure. Since the generalization from objectual to propositional structures is such that the only formal feature of a proposition is its binary value (true or false), a propositional operator is logical iff it is invariant under 1-1 mappings of propositions which transfer each proposition into a proposition with the same binary value (i.e., its truth value).

Technically, we can define:

(a) A propositional structure is an \( n \)-place tuple \( \langle P, p_1, \ldots, p_n \rangle \), where \( P \) is the set of all propositions of a given language and \( p_1, \ldots, p_n \) are elements of \( P \).

(b) An argument-structure for a \( k \)-place propositional operator is a propositional structure of length \( n + 1 \).

(c) Two propositional structures \( \langle P, p_1, \ldots, p_n \rangle \) and \( \langle P, p'_1, \ldots, p'_n \rangle \) are isomorphic if \( n = m \) and there is a 1-1 truth-preserving function \( f \) from \( P \) onto \( P' \) such that for every \( p \in P \), \( p' = f(p) \) is the image of \( p \) under \( f \).
Truth-functionality is thus (logical) formality on the propositional level.\textsuperscript{18} We are now ready to consider Feferman's criticisms.

**FEFERMAN'S CRITICISMS**

The Invariance-under-Isomorphism criterion is a substantive criterion and as such it invites substantive criticism. In "Logic, Logic and Logic" (1995), Solomon Feferman offers three substantive criticisms of the claim that the criterion is a necessary and sufficient criterion of logicality (referred to as the "the Tarski–Sheffer thesis"). Feferman formulates the criticism (in terms sanctioned by a certain deflatability result due to McGee 1996) as follows:

An operation $O$ across domains is a logical operation according to the Tarski–Sheffer thesis if and only if for each cardinal $k$, if $k$ is a formula $\alpha_k$ of $\mathcal{L}_k$, which describes the action of $O$ on domains of cardinality $k$.

(Feferman 1999: 37)

Here, however, I will continue to employ our earliest terminology in discussing his criticisms.

Feferman criticizes the Tarski–Sheffer thesis on three counts:

1. "The theorem announced is too strong, more specifically in set theory. (Ibid., my italics).

**Elaboration:**

The first (purely), I think, speaks for itself...but it will evidently depend on one's gas, that is, one's grasp of the nature of logic as to whether this is sound or not. For Sheffer, to take one example, this is no problem. Indeed, the thing that he is bound to in any way is not the concept of logical meaning. Any higher-order mathematical predication or relation can function as a logical term, provided it is introduced in the right way into the syntactic-semantic apparatus of first-order logic." (Shieh 1991, p. 295–96) What that "right way" is for him is spelled out in a number of syntactic-semantical conditions... (Ibid., pp. 34–37) "Although there are conditions which are not logical operations... of type-level at least 2... (Shieh 1991, p. 36) there is no characterization criterion... in particular, we can express the Consistency Hypothesis and other substantial mathematical propositions as legitimate determinations of the Tarski–Sheffer thesis... But in so far as... she denies the existence of an inferential relation of a special kind, or at least of that determinative property, it is evident that we have not the same conception of logic as the sense of universal notions independent of \"what there is.\"

(Feferman 1999: 37–38)

2. The theorem announced is too strong in explaining the semantics of the background language on the metalanguage. (Ibid., my italics)

There are of course many familiar ways to construct isomorphisms, for example, using structures whose distinguished elements are truth-values. But what I was looking for is a construal that would be philosophically transparent, regardless of its finitistic or infinitistic.

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Elaboration:

Point 3 is in a way subsidiary to point 1. The notion of \"invariance\" for an-axiomatic theories can be vague, but the idea is that if logical notions are as all be explained on an-axiomatically, they should have the same meaning independent of the next context of the axiomatic universe. For example, they should give equivalent truth to the constructive set and in functioning contexts. Godel's well-known concept of shuffling provides a necessary criterion for such notions and, when applied to the kind of operations considered by the Tarski–Sheffer thesis, immediately reduces that same thesis. For example, the quantifier \"there exist uncountably many\" would not be logical according to this criterion, since the property of being uncountable is not definable.

(Frederick 1999: 38–39)

Feferman, however, qualifies his support of the absoluteness criterion somewhat: Our theory is aware that the notion of absoluteness is self-reflexive and is sensitive to a background set theory, because it is the question of what notions exist.

( Ibid.)

3. "The natural operations are given by the Tarski–Sheffer Thesis of what constitutes the same logical operation upon arbitrary base domains." (Ibid., my italics)

Elaboration:

It seems to me there is a sense in which the usual operations of the first-order predicate calculus have the same meaning independently of the domains of individuals over which they are applied. This characterization is not captured by invariance under bijections. As McGehee put it, "The Tarski–Sheffer thesis does not require that there be any connections among the ways a logical operation can be defined over domains of different sizes. Thus, it would permit a logical connection which act like disjunction when the size of the domain is an even number cardinal, like conjunction when the size of the domain is an odd number cardinal, and like a biconditional at limits." (McGehee 1996: 367)

(Feferman 1999: 39)

For Feferman, this point is more compelling than the other two:

For me, point 3 is perhaps the most important for rethinking the Tarski–Sheffer thesis or least as it stands.

(Frederick 1999: 39)

But his objection concerns only the sufficiency part of the Tarski–Sheffer thesis:

I agree completely [that] the Tarski–Sheffer thesis is a necessary condition for an operation to count as logical.

(Feferman, invented sentence statement)

Still, it is a clear and strong criticism.

I... believe that if there is to be an explanation of the notion of a logical operation in semantic terms, it has to be one which shows how the way an operation behaves when applied over one domain $D_1$ connects naturally with how it behaves over any other domain $D_2$.

(Feferman 1999: 38–39)
As a "first step in that direction," Feferman presents a revision of the Invariance-to-
sets-homomorphism criterion. The revision centers in replacing "homomorphism" by "Homotopyism," the resulting concept of logical equivalence being that of a "homotopy equivalent invariant operation" (ibid. 9). I will examine Feferman’s proposal below, but first let me consider his criticisms.

CONSIDERATION OF FEFERMAN’S CRITICISMS

I will begin by putting Feferman’s criticisms in a proper perspective. There are a few significant points of similarity between Feferman’s approach to the issues in question and mine (I prefer not to speculate about Tait’s). First, Feferman does not question either the need for a criterion of logicality or the appropriateness of the semantic method for such a criterion. Second, Feferman regards the issue of logic as a foundational issue, and it is not even the prototypical foundational studies. On the contrary, Feferman has been vociferously engaged in important foundational work, e.g. examples of which are Feferman 1996a and Feferman and Hellman 2000. Furthermore, Feferman’s approach is indeed logicist, not Platonist, or antirealist, or intuitionist, or indispensability, but it is setting a new approach to the foundations of logic and mathematics. (See, e.g., Feferman 1984, 1993a, and 1995a.) Finally, Feferman, as noted above, accepts the Invariance-under-homomorphism criterion as it stands as an elementary condition on logicality and its own proposal for a sufficient condition on logicality only a limited version of that criterion. In light of these observations, I think it is reasonable to view Feferman’s criticism as a more general, more direct criticism, rather than a full-scale antilogical criticism. Nevertheless, this is a very significant criticism that requires careful consideration.

Assimilation of logic to mathematics

A disagreement between a mathematician and a philosopher on the relation between logic and mathematics, such as that between Feferman and myself, was anticipated by Tait:

[This is a point very much in the question whether mathematics is separate from logic or is a different kind of logic. A nominal conception of logic, like theory and mathematics, appears, I think, to be a natural tendency of modern philosophy. Mathematics, on one hand, would be disposed to treat the mathematics, which they consider the higher discipline in the world, as a part of something we regard as logic and they therefore provide a development of set theory in which set-theoretical objects are not logical objects.]

(Tait 1966, 959)

The elliptic and square-bracket formulations are partly intended to parastat Tait’s tendency to identify the question of the logical characters of operation criteria on logic and mathematics with logicalism. As noted in his paper in this volume, his own view is that the "reformulation" of logic which he is in the "categorization" of mathematics, although

But I think there is more to Feferman’s position than a certain type of misad of a ver-

tinal attitude toward mathematics. In my view (as outlined above) mathematicians have a valid reason for regarding mathematics as dealing with non-logical concepts, namely their task. Their task is to discover and systematize the laws governing formal structures rather than apply these laws in discovering and reasoning. And the natural way for humans to study the laws governing a certain kind of structure is to construct the mathematical structures as structures of basic elements (of some kind), e.g., in the case of formal structures as structures of elements that do not violate the Invariance-under-homomorphism criterion. But the two conceptions of formal objects do not conflict. To see this more clearly, let us draw on my own to the conception of numbers in mathematical structuralism.

From the structuralist point of view there is no difference between studying the laws of arithmetic by studying a certain system of numbers or the corresponding systems of sets. But to study the arithmetic laws the mathematician is best served by choosing some specific entities to work with, e.g., the natural numbers as set. From the point of view of the working number theorist, then, arithmetic is a theory of a specific kind of object, but that does not conflict with the philosophical claim, made by abstraction and generalization, that numbers are mere objects or structures, whose occupants’ identity is immaterial. In the case of formal systems, the notion that mathematicians work with are, for the most part, lower-order, non-logical systems, while the notion that logicians work with are for the most part, logical systems, obtained from lower-order, non-logical, mathematical systems by "raising" them to a higher-order. It is this view that captures our nature as formal or structural elements, and the laws governing them as laws of formal structure. Together, these two perspectives give us insight into the nature of formalism. So we see that Feferman’s justified claim that there are significant differences between logic and mathematics is in fact satisfied by the Invariance-under-

homomorphism criterion, especially on the formalist interpretation I have given to it here and in (t91). Feferman’s criticism, however, runs on other issues as well, some directly, others indirectly. One issue is raised indirectly by the example of common sense, or what I call "sure feeling," in determining the relation between logic and mathematics. On this issue, I am aware, we are in disagreement, since in my view the relation between logic and mathematics has very little to do with sure feeling. It is one that in approaching this issue, and in various ways of giving it, we use everyday intuition. But once we approach it as a theoretical issue, as we do when we construct a rigorous criterion of either way mathematics and logic are one, the deciding-in of mathematics’ significant logics is metaphysics in mathematics the properties of objects associated with logic, while mathematicians’ attitudes to the properties are usually associated with mathematicians.

An analogy with the concept of class in mathematics might help. In some sense a group is better expressed by its equivalence class than by any of its constituents. But an equivalence class can represent the class without its constituent classes, which are generally not equivalence classes, exemplifying it. In that sense, there is a division of labor between equivalence

and non-equivalence classes in expressing the idea. This, indeed, is a natural way to understand the mathematical structures as well.
logicality and develop a systematic account of logic to go with it, the role of gun-feelings becomes very limited. In fact, Feferman himself regards foundational studies as having a largely theoretical role only—"conceptual clarification; interpretation (and) reduction... of problematic concepts and p-angles; organizational foundations; and effective expansions of concepts and principles"—(Feferman 1958:10). As such they are "entailed to results that conflict with some of our gun-feelings." Since the Invariance-under-isomorphism criteria, combined with the formalist account of logic, offers an informative and systematic account of the concept of logical operator, these second serious conceptual problems (e.g., with the definition of logical consequence) require the relation between logic and truth. Feferman stresses the role of logic in our system of knowledge, critically evaluates many of the incorrect attributes of logic, and offers a substantive and methodologically systematic account of the relation between logic and mathematics, it should not be judged based on "gun feeling."

Another issue raised by Feferman's criticism is onomological commitments. Feferman underlines the traditional view that logic, unlike mathematics, should have no ontological commitments. By asuming logic to be mathematics, he claims, the Invariance-under-isomorphism criteria burden it with un-welcome ontic commitments. By this Feferman means one of two things: (i) the fact that we resort to a set theoretical background language carries with it ontological commitments to sets; (ii) the enormous expressive power of the logical sanction by that criterion carries commitments to many ontologically laden set theoretical theories. Clearly (i) is contrary to standard first-order logic and the logic sanctioned by our criteria. So let us turn to (ii). Consider the sentence:

$$(\exists x)(x=x \land \forall y y=y)$$

It is a well-formed sentence of the logic sanctioned by the Invariance-under-isomorphism criteria, but for its truth-value in uncountable models to be determined, logic must be committed either to the continuum hypothesis (CH) or to its negation (¬CH). This does not model logic with the same onomological commitments as those of mathematics. To get a first inkling of the difference between logical and mathematical commitments, consider the difference between the way the logical CH and the mathematical CH behave under negation. (This is a theme known from comparisons of first- and second-order CH; see, e.g., Shapiro 1991.) Let us call the mathematical statement expressing CH "CH_2" and the logical statement expressing CH "CH_1." Then, whereas "CH_2" is captured by "¬CH_1," it is not captured by "¬CH_2." "¬CH_2" can be added to set theory at all axioms without rendering set theory inconsistent. But "¬CH_1," cannot be a logical law, since logic—both standard logic and the logic sanctioned by the Invariance-under-isomorphism criteria—has countable models (in which CH is trivially satisfied), and these would prevent it from being true in all models.

The main point is that while mathematics has direct ontological commitments, logic's ontological commitments are for the most part indirect. Aside from a few direct technical commitments—for example, a commitment to the existence of at least one individual (given the technical requirement that a model have a non-empty universe)—logic has only indirect ontological commitments, namely, commitments through its background theory of formal structure. And even these commitments are not existent in the usual sense; rather, they are commitments to the formal possibility of existence. Thus, as an analogy with (mathematical) set theory, Infinity says that an infinite set actually exists, but as an axiom within a background theory for logic, it says that an infinite structure of objects is formally possible.12

Non-robust logical notions

Feferman notes that many of the logical operators sanctioned by the Invariance-under-isomorphism criterion are not "robust" and argues that only "robust" operators should be classified as logical. The word "robust" can be interpreted in many ways, but Feferman has a specific interpretation in mind for "robust" defined in set-theoretical terms that is robust in the sense of having a "the same meaning independently of the exact existence of the set-theoretical universe" (quoted above). And this idea, Feferman suggests, is captured by the set-theoretical concept of "absolutes," to be robust is to be "absolute" in the set-theoretical sense. The set-theoretic concept of absolute was introduced by Gödel in the course of proving the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis. His proofs involved the claim that in the "constructible universe" V=L (i.e., the whole universe of set), and to establish this claim Gödel used absoluteness results, whose basic concepts can be defined as follows: A formula $\varphi(x_1, \ldots, x_n)$ is absolute from a transitive class M as a transitive admissible N iff $\forall x_1 \ldots \forall x_n [\varphi(x_1, \ldots, x_n) \in N \rightarrow \Phi = M \rightarrow \Phi]$.

Gödel was especially interested in formulas which are absolute from V to L and, in particular, in the fact that the operation of forming all the "constructibles" (definable) subsets of a given set is absolute from V to L. (See Gödel 1940 and discussion in Solovay 1990.) But the concept of absoluteness has been generalized in various ways, leading to many new applications.13 From the point of view of Feferman's criticism of the Invariance-under-isomorphism criteria, the most relevant feature of the absoluteness requirement is that it does not allow operators to change their meaning by separation or contraction of a given universe. This requirement renders "Gödel's absoluteness operator" (relative to ZFC) but "unsusceptible" not a subset of the universe that satisfies "is finite" in a smaller

12 I briefly discussed this issue in Sher (1994c:682) where I refer to other references. Note: I do not mean to say that including CH, for example, in an axiom of our background theory of logic does not violate canonical theory to provide the functions similar to accept the fact that the size of the continuum, is given). What I mean to say is that if CH is included in an axiom of this theory, then it represents a formal law whose scope is the result of formally possible structures of objects. If we include CH in this theory, we are not inherently committed to the possibility of constructing an object for something that will do the same job. I would like to think that I have made an error in saying the same.

model of set theory also satisfies it in a larger model (and vice versa, assuming it is included in the smaller model), but a universe of the latter kind that satisfies it is uncountable in a small model (for example, a Lowenheim–Skolem model) does not satisfy it in a standard model. Accordingly, the quantifier "infinitely many" is absolute, but "uncountably many" is not. But both quantifiers are logical according to the Lowenheim–Skolem criterion, therefore, this criterion must be rejected, or so Feferman says.

In response to Feferman’s second criticism, I will first show that this criticism is weaker than it may seem to be, and then I will question the relevance of absoluteness to logic.

(A) First, it should be pointed out that Feferman’s criticism is directed at an aspect of particular background theory we use to formulate the Invariance-under-Automorphism criterion, but the idea underlying the criticism is not wedded to this, or any other background theory. In particular, the conception of logicality as formula- and non-conceptual notion of finiteness under 1–1 replacements of individuals, is not inherently connected to a particular set-theoretical language for which the question of “absoluteness” arises.

But even assuming this background language, Feferman’s criticism is weaker than it may seem to be. For whereas one can see the operators “uncountably many” changes its meaning from universe to universe, in another, more relevant sense, it does not. Let me explain. Clearly, as defined in a first-order set-theory, call it “V,” the predicate “x is uncountable” is satisfied by some countable set (i.e., an individual y to which countably many individuals a stand in the relation “a is a member of y”) in some model of V. But the quantifier “there are uncountably many” is not satisfied by any countable set (a collection of countably many individuals) in any model of a first-order logical system in which it serves as a logical quantifier. To see this, the reader has to know how such a logical system is constructed, and this is something I have not discussed here. (A relevant discussion appears in chapter 3 of Shel 1991.) But let me try to explain the general principle underlying this claim briefly.

Consider the following: In a first-order set-theory, T, we cannot see that the predicate “x is uncountable” of T is satisfied by a countable set in some model of T. To see that this is so, we have to go to another theory, T2, which is at least as strong (in the relevant sense) as T, and in which we can truly say that the formula “x is uncountable” of T is satisfied by some countable set in some model of T. Intuitively, from the point of view of T, the predicate “x is uncountable” is not robust, but from the point of view of T2, it is.

Now, it is a remarkable feature of a logic L, that the following are all done on the same level of discourse, within the same background theory—call it “T1.” (i) the definition of the logical constants of L, (ii) the definition of the operation corresponding to

\[ \text{Ie., consistent from the point of view of } T_1. \]
need to renounce the formal conception of logic and the associated justification of our criterion of logicality. I would be very interested to examine a philosophical conception of logical truth that fits in with the demands requirement and offers a foundational justification of a concept of logicality satisfying it. As far as I know, none is available yet.

Operations with "non-uniform meaning", "split identity", or "unnatural behavior".

Feferman’s main objection to the invariance-under-isomorphism criterion is that it sanctions logical operators lacking a unified identity, or a natural connection between the way they behave in different universes, or (when we consider the terms denoting them) the same meaning in different universes. Such operators can behave one way over universes of cardinality \( \kappa \) and another way over universes of cardinality \( \lambda \neq \kappa \), i.e., their meaning, or identity, depends on the size of the universe, and there is no natural connection between the way they behave in universes of size \( \kappa \) and universes of size \( \lambda \).

Before considering Feferman’s criticism, it would be instructive to note that his particular example of such an operator is in fact not encountered by my version of the “Turksir” thesis. Feferman’s example is that of a propositional connective, \( O \), “which acts like a disjunction when the size of the domain is an odd successor cardinal, and like a biconditional in finite” (cited above). I agree with Feferman’s point, that \( O \) is not a proper logical operator, but not with his reason for claiming so. Propositional connectives should not depend on the size of the universe (of individuals) because this has nothing to do with meaningfulness. The problem with \( O \) as I see it, is that as a propositional operator it should not take as input universes of individuals at all. And in my version of the logicality criterion propositional operators do not. Propositional operators (connectives) are defined in terms of propositional rather than objectual structures, and propositional structures have a universe of propositions rather than a universe of individuals. Indeed, they take as input only one universe — the universe of all propositions. The operator mentioned to Feferman’s example is therefore not logical according to my version of the “Turksir” logicality criterion.

But the phenomenon Feferman talks about is one of other operators violating this criterion. Take the objectual operator \( Q \) defined by Feferman’s universal \( A \) and subset \( B \) of \( A \).

\[ Q(B) = \{ x \in A : x \text{ is countable and } B \cap x \neq \emptyset \} \]

Then \( Q \) behaves like \( \emptyset \) in countable universes and like \( \emptyset \) in countable universes.\footnote{This is not the only kind of operator that exhibits this phenomenon. In connection with my claim that propositional connectives are not sensitive to the size of objectual universes, I would like to clarify that some logical operators defined in terms of objectual structures corresponding to non-logico-structural notions (orders, contours, etc.) are sensitive to the size of universes of individuals, but their objectual definitions do not correspond to any propositional connectives. Thus, the functional operator \( \mathbf{F} \), defined for an objectual universe \( A \) and subset \( B \) of \( A \), is}

\[ \mathbf{F}(B) = B \cup \mathbf{F}(A \setminus B) \]

is a logical operator according to the invariance-under-isomorphism criterion, but it is not the objectual analogue of a propositional connective.

(Turksir’s Thesis)

Is the fact that \( Q \) has the “split” identity a good reason for refusing to count it as a logical operator? In answering the question I will make a few points:

(A) To the extent that we refuse to count \( Q \) as a logical operator because it is an “unnatural” operator, it should be noted that numerous unnatural object properties (relations, functions, etc.) are widely accepted in other fields. Feferman himself (2000) brings numerous examples of what he calls “metaphysical” or “pathological” objects that are generally accepted by mathematicians (he included).

Indeed, even in standard logic there are many “unnatural” operators, including logical operators of “split” identity or meaning, operators which do not seem to have the same meaning or be the same operators in different settings. Two examples would suffice:}

\[ (a) \quad \text{A 132-place propositional \( \land \)-successor, \( C \), such that:} \]

\[ (i) \quad C \text{ behaves like a 132-place conjunction in rows with } 3 - 25 \text{'s}, \]

\[ (ii) \quad C \text{ behaves like a 132-place disjunction in rows with } 26 - 79 \text{'s}, \]

\[ (iii) \quad C \text{ behaves like the majority connective in all other rows.} \]

\[ (b) \quad A \text{ quantifier \( Q \), definable in standard first-order logic, such that:} \]

\[ (i) \quad Q \text{ behaves like } \exists! \text{ (1) in universes of cardinality } 101, \]

\[ (ii) \quad Q \text{ behaves like } \exists! \text{ (2) in universes of cardinality } 101, 745, \text{ and} \]

\[ (iii) \quad Q \text{ behaves like } \emptyset \text{ (3) in universes of all other cardinalities.} \]

Intuitively, the identity (or meaning) of these operators is no less “split” than that of \( Q \); they are accepted as legitimate logical operators by most logicians and philosophers (including Feferman, I’m sure). Why, then, should we discriminate against \( Q \)? In what way is \( Q \) unnatural, or its identity or meaning more “split”, than those of \( C \) and \( Q \), which we accept as legitimate logical operators?

(B) My point is not that there is no value in insisting on a specific concept of “natural” operator or “natural connection between an operator’s behavior in different universes” but that such a concept has nothing to do with our idea of logicality.

We may wish to distinguish “natural” logical operators from “unnatural” logical operators or natural operators in general from unnatural operators in general,

\[ [2] \quad \text{I should indicate that I had the opportunity to consult Feferman’s errors when I received a prepublication copy of McGee’s paper (1980) from which the example above, but I failed to do so, since in the errata of McGee’s paper it was noted in a note that, in the present context, however, it is more significant.} \]
in the same way that we may wish to distinguish "natural" functions from "unnatural" functions or "natural" relations in general from "unnatural" relations in general. But just as the latter would not undermine, or force us to change, our criterion of a functionality of a relation being functional, so the former would not undermine, or force us to change, our criterion of a logical system being logical.

We could impose on ourselves a "naturalness" constraint in choosing a logical system to work with, but this would be a separate constraint from the "logicality" constraint we would impose on such a system.

Finally, there is a strong entry (or similarity) to the one-up of logical operators delineated by the invariance-under-homeomorphism criterion and a "true concept of a logical system" associated with it. Both are generated by the interpretation of this criterion as a criterion of formality. All and only formal operators are logical, and each logical operator describes one way in which an operator takes the account of some formal features of a given situation. Thus all logical operators are universal in at least one of the universes if there is some formal pattern of objects having properties and standing in relations within that pattern through the different universes in question. Since the universe of the world is a basic formal feature of philosophical situations, it is—and should be—a central parameter of any objective formal system.

**SFlatman’s Criterion of Logicality**

SFlatten's criterion of logicality is proposed "as a first step" in the "direct line" of showing "how the way an operation behaves when applied over one domain is connected naturally with how it behaves over any other domain." (SFlatten 1989: 3a–39.) It is obtained from the Lewy–Schild–SFlatten criterion by replacing "isomorphic" by "homeomorphic" (as what is sometimes called "strong homomorphism"). This is, by replacing the requirement that logical operators be invariant under any homeomorphism with a weaker condition that they be invariant under any homeomorphism.

**Invariance-under-Homeomorphism:** An operator O is a logical if it is invariant under all homeomorphisms of its argument structures.

where:

(i) A structure \( (A, \varphi_1, \ldots, \varphi_n) \) is homeomorphic to a structure \( (A', \varphi'_1, \ldots, \varphi'_n) \) if \( n = k \) and there is a bijection \( f \) from \( A \) to \( A' \) such that for every \( 1 \leq i \leq n \), \( \varphi_i \) is the image of \( \varphi'_i \) under \( f \).

(ii) An n-place operator \( O \) is invariant under all homeomorphisms of its argument structures if for any of its argument-structures, \( (A, \varphi_1, \ldots, \varphi_n) \) and \( (A', \varphi'_1, \ldots, \varphi'_n) \), if \( (A, \varphi_1, \ldots, \varphi_n) \) is homeomorphic to \( (A', \varphi'_1, \ldots, \varphi'_n) \), then \( O_\varphi(A, \varphi_1, \ldots, \varphi_n) = O_{\varphi'}(A', \varphi'_1, \ldots, \varphi'_n) \).

The effects on the generality and formality of our concept of logicality are (i) since every bijection is a surjection but not vice versa, there are more surjections than bijections, and Invariance-under-Homeomorphism (satisfaction) is invariance under more transformations of structures. As a result, the concept of logicality associated with the new criterion is more general than that associated with the old criterion. All logical operators under the former are logical under the latter, but not vice versa. The new criterion, however, does not mean logic in a logical system to work with, but this would be a separate constraint from the "logicality" constraint we would impose on such a system.

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not with our “formal” logic, yet it is not intermediate between the two either, since in certain ways it is weaker than standard first-order logic. In particular, neither the identity relation nor the finite-cardinality quantifiers are “fundamental” in the sense that the identity relation is not a primitive feature of the language, and the finite-cardinality quantifiers are not ones that can be expressed by a single quantifier.

Felsenstein does not fully embrace the Invariance-Under-Homomorphism criterion as a criterion of logicality. Rather, having been moving more and more to the position that the classical first-order predicate logic has a privileged role in our thought (ibid., 12), he is looking for ways to adjust it as an intransitive and —as the standard logical operations are—truth-functional. His investigations first lead to an adjustment that, assuming Invariance-Under-Homomorphism is unformulated as it is to apply to functional operations only, could be expressed by:

**Adjusted Invariance-Under-Homomorphism criterion (I):**

A first-order operator is logical if it is:

- (i) a monadic quantifier satisfying the Invariance-Under-Homomorphism criterion,
- (ii) a truth-functional connective,
- (iii) a relation definable from logical operators within the \( \lambda \)-calculus.

By formulating the Invariance-Under-Homomorphism criterion in such a way that it applies to propositional connectives as well, however, Felsenstein obtains a more unified version of this adjusted criterion:

**Adjusted Invariance-Under-Homomorphism criterion (II):**

A first-order operator is logical if it is:

- (i) a monadic quantifier satisfying the Invariance-Under-Homomorphism criterion,
- (ii) a propositional operator (monadic or non-monadic) satisfying the criterion,
- (iii) a relation definable from logical operators within the \( \lambda \)-calculus.

The adjusted criteria differ from the original Invariance-Under-Homomorphism criterion in that it is not a prerequisite on logical quantifiers: only monadic first-order quantifiers—quantifiers of the type \( Q X \phi \), where \( X \) is a term of a given universe—and not first-order quantifiers of any other type—i.e., relational or polyadic quantifiers—are logical. That is, only monadic quantifiers are subject to the Invariance-Under-Homomorphism test. (Linguistically, this restricts us to quantifiers of the form “\( Q x \phi \)”, ruling out in advance, i.e., prior to applying the Invariance-Under-Homomorphism criterion, all relational quantifiers (e.g., “\( \forall x \phi \)”, “\( \exists x \phi \)”, and “\( \forall x \exists x \phi \)”)).

This restriction yields the desired result: all and only the logical operators of standard first-order logic achieve identity are logical.

What about identity? By considering this question, Felsenstein says:

It is understandable that the relation of identity has a “universal”, accepted, and basic role in the presence of usually defined predicates and functions, as is usual in PC with \( \forall \), and

**Tarski’s thesis**

that argues for giving a distinguished role in logic even if it should not turn out to be logical on its own under some quasi-ontological (invariance criterion, such as) homomorphism.

(1954: 46)

To include identity as a logical operator we can simply postulate that it is closing logical operators under definability as before. We thus get the third version of the adjusted criterion:

**Adjusted Invariance-Under-Homomorphism criterion (III):**

A first-order operator is logical if it is:

- (i) a monadic quantifier satisfying the Invariance-Under-Homomorphism criterion,
- (ii) a propositional operator satisfying the criterion,
- (iii) the identity relation,
- (iv) an operator definable from logical operators within the \( \lambda \)-calculus.

This criterion classifies identity and the finite-cardinality quantifiers as logical, thus providing a characterization of the standard first-order logical operators as logical. Is either the Invariance-Under-Homomorphism criterion or the Adjusted Invariance-Under-Homomorphism criterion (or any of its versions) an adequate criterion of logicality?

Van Benthem (2002) and Bonnay (2008) point out that the Invariance-Under-Homomorphism criterion is subject to Felsenstein’s first two criteria—assimilation of logic to mathematics and non-robot logical operators—and as such is inadequately from his own perspective. I would add to that by affirming the logicality of the finite-cardinality quantifiers—including “\( \forall x \phi \) identity/mapping” finite-cardinality quantifiers (those whose behaviour at convergence of different sizes is “causally sufficient”)—Felsenstein’s third condition also violates the third criterion. Finally, Bonnay (2008) criticizes the ad hoc range of Felsenstein’s restriction of logical quantifiers to monadic ones in the adjusted version of his criterion. 52

Most of these criticisms, however, do not speak against Felsenstein’s criteria from my point of view, since the “weakness” they refer to arises weaknesses at all from my perspective. The one exception is the ad hoc range criticism, which points to what, in my view, is the main challenge to any criterion of logicality, namely, a solid philosophical justification, which is missing from Felsenstein’s discussion, and indeed not even attempted by him. That such a justification needs pursuing is also Felsenstein’s view of the matter:

Whether this (i.e., the notion of a logical operator as “definable from homomorphism-invariant monadic operators”) for any other invariance notion) can be justified on fundamental conceptual grounds is in need of pursuit.

(1959: 32)

Felsenstein’s aim to justify this restriction linguistically, by appealing to a linguistic contention which relies on some monadic quantifiers used in natural language are “definite” in one way or another (e.g., monadic quantifiers (Koerner and Mouton, 1999). But this contention is raised unjustifiably and is based on a different version of Invariance-Under-Homomorphism. The discussion above also assumes the logicality of monadic quantifiers that Felsenstein rejects. More importantly, it is not clear that linguistic support of a logical philosophical restriction is of much relevance.

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This is a good move on which to end. I must add, however, that there seems ever serious proposals for revision of the Ultrametius-under-homomorphism criterion. These include Przewlocki (1976), McCarthy (1981), MacFarlane (1993), Baroni (2008), and Castronova’s (2007), and they each require a careful consideration.

References


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