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# Tarski's Thesis

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"Tarski's Thesis" is the claim that a certain invariance condition can serve as our criterion of logicality. My goal in this chapter is to explain the thesis, provide it with a philosophical justification, and respond to three recent criticisms due to Solomon Feferman.

## CRITERION OF LOGICALITY

In a 1966 lecture, "What are the Logical Notions?", Tarski's proposed the following criterion of logicality:

**Invariance under Permutation:** A notion is logical iff it is invariant under all permutations of the individuals in the "world" (or universe of discourse).[1]

By "notions" Tarski understood not linguistic or conceptual entities but objects of the kind referred to by such entities, i.e., objects in the world, including individuals, properties (sets), relations, and functions. "World" he understood as including both physical and mathematical objects and as forming a type-theoretic hierarchy, based on *Principia Mathematica* or a similar theory. In the present context it will sometimes be convenient to view objects as operators (characteristic functions representing them) and use standard set theory with urelements rather than *Principia Mathematica* as our background theory.

By centering his attention on objects or operators (worldly entities) rather than constants (linguistic entities) Tarski follows the precedent of the Boolean, truth-functional definition of logical connectives in propositional logic. This definition

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[1] Paraphrase of Tarski (1966): 149.





identifies logical connectives with certain objects, namely, Boolean truth functions, and it is these objects, rather the names or descriptions used to refer to them, that are said to capture the idea of logicality on the propositional level. One advantage of the objectual route is that it avoids complications arising from the vagaries of linguistic usage.[2] Another advantage is the existence of a richer, more precise, and more sophisticated machinery for talking about operators than about constants.

Before examining Tarski's specific criterion, let us consider the idea of a general criterion of logicality independently of its content. What is the purpose of such a criterion? What would a systematic principle that demarcates the logical from the non-logical (not just on the level of propositional connectives but also on the level of quantifiers and other non-propositional operators) accomplish? The answer, I believe, is this: First, it would bring an end to the current practice of an ad-hoc, utterly uninformative, definition-by-enumeration of the logical operators other than connectives. Second, it would solve a serious problem that threatens to undermine Tarski's model-theoretic definition of logical consequence, and with it the entire field of logical semantics. Furthermore, such a principle would considerably deepen our understanding of the nature of logic, expand our ability to approach logic critically, create a fertile domain of mathematical investigations, help solve outstanding problems in linguistic semantics, and perhaps make other contributions as well, e.g., explain the relationship between the concept of logicality and other central philosophical concepts, explain logic's relation to neighboring fields (both within and outside philosophy), and so on.

One would have expected Tarski to motivate his criterion by the problem that threatened his own definition of "logical consequence," and whose full import he recognized and brought to our attention (Tarski 1936), namely, the problem that the definition's adequacy depended on the existence of an adequate criterion of logicality. At the time Tarski worried that such a criterion would never be found (in which case his definition would be forever unjustified), and this naturally leads us to expect that his 1966 lecture was intended to assuage those worries.

However, judging from what Tarski explicitly said (and did not say) in his 1966 lecture, his route to the criterion of logicality was completely divorced from his early concerns.[3] Instead, Tarski arrived at this criterion based on general considerations concerning the demarcation of fields of knowledge. His starting point was Klein's demarcation of geometrical fields based on their invariance properties. Klein suggested that each geometric field could be characterized by the invariance condition satisfied by its notions. This condition had the form:

**Geometric Invariance:** Geometric notion O is invariant under all 1 – 1 transformations of the geometrical space onto itself which preserve X.

[2] See Sher (2003). A similar advantage accrues to the objectual, model-theoretic definition of logical consequence as opposed to the linguistic, substitutional definition of this concept. (See Tarski 1936 and Sher 1996*a*.)

[3] One of the things that Tarski explicitly said (p. 145) is that he was *not* interested in the problem of logical consequence (or, as he put it, logical truth) in that lecture.

By strengthening X we restrict the transformations taken into account, getting more specific geometrical notions; by weakening X we increase the transformations taken into account, getting more general notions. Thus, if X is the requirement that the ratio of distances between points be preserved, the class of notions satisfying Geometric Invariance is the class of *Euclidean* notions. By strengthening X to the requirement that actual distances between points be preserved we obtain a characterization of narrower geometric notions, namely those applicable to *rigid bodies* (which don't change their *size* under movements or transformations); and by weakening X to the requirement that (I will express by saying that) openness (open sets) be preserved, we obtain a characterization of very broad geometric notions, namely the *topological* notions. Now, Tarski asked: What would happen if we weakened X as much as possible, i.e., if we set no requirements on the transformations taken into account? Then, we would get the condition.

**General Invariance:** Notion O is invariant under all 1–1 transformations of space, or the universe of discourse, or the "world" onto itself (or under all permutations of the "world").

This invariance condition takes *all* 1–1 transformations into account and, as a result, characterizes our most general notions. What is the science which studies these notions? Tarski suggested that this science is *logic*. Logic deals with our most general notions, notions which are invariant under all 1–1 transformations of the world onto itself.

Today, we usually adopt a slightly different version of Tarski's criterion. In fact, Tarski's (1966) lecture remained unknown for many years, and the current version is historically traced to Lindström's (1966) generalization of Mostowski (1957). This version invokes "isomorphisms" (or "bijections") instead of "permutations" (or "transformations") and refers to a totality "structures" rather than a to single, universal, "world." One way to formulate this criterion is:

**Invariance under Isomorphism:** An operator O is logical iff it is invariant under all isomorphisms of its argument-structures

where:

(i) A *structure* is an m-tuple, $m \geq 1$, whose first element is a universe, A (i.e., a non-empty set of objects treated as individuals, that is, as objects lacking inner structure), and whose other elements (if any) are set-theoretic constructs of elements of A.

(ii) Two structures, $\langle A, \beta_1, \ldots, \beta_n \rangle$ and $\langle A', \beta'_1, \ldots, \beta'_k \rangle$, are *isomorphic*—$\langle A, \beta_1, \ldots, \beta_n \rangle \cong \langle A', \beta'_1, \ldots, \beta'_k \rangle$—iff $n = k$ and there is a bijection $f$ from A to A′ such that for every $1 \leq i \leq n$, $\beta'_i$ is the image of $\beta_i$ under $f$.

(iii) An operator O represents an *object of a given type*—an individual, a property of individuals, an n-place relation of individuals ($n > 1$), an n-place function from individuals to an individual, a property of properties of individuals (i.e., a monadic first-order quantifier), a relation of properties of individuals (i.e., a relational first-order quantifier), a property of relations of individuals (i.e., a polyadic quantifier), etc.—and specifies its extension (or constitution) in each universe.



Specifically:

- An operator representing an individual $a$ assigns to each universe A a 0-place function whose fixed value is $a$ if $a \in A$, and which is treated in some conventional manner otherwise.
- An operator representing a first-order property assigns to each universe A a function from all members of A to a truth-value (which, provisionally, we assume is T or F).
- An operator representing an n-place first-order relation ($n > 1$) assigns to each universe A a function from all n-tuples of members of A to {T, F}.
- An operator representing a first-order monadic quantifier assigns to each universe A a function from all subsets of A to {T, F}.
- An operator representing a first-order binary relational quantifier assigns to each universe A a function from all pairs of subsets of A to {T, F}.
- An operator representing a first-order polyadic quantifier (of the simplest type) assigns to each universe A a function from all binary relations on A to {T, F}.

Etc.

(iv) If O is an operator whose arguments are of types $t_1, \ldots, t_n$,[4] A is a universe and $\beta_1, \ldots, \beta_n$ are constructs of elements of A of types $t_1, \ldots, t_n$ respectively, then $\beta_1, \ldots, \beta_n$ are arguments of O in A (or $\langle \beta_1, \ldots, \beta_n \rangle$ is an argument of O in A) and $\langle A, \beta_1, \ldots, \beta_n \rangle$ is an *argument-structure* of O.

For example:

(a) The first-order property "is red" is represented by an operator, R, which for every universe A is assigned a function, $R_A: A \rightarrow \{T, F\}$, such that for any $a \in A$, $R_A(a) = T$ iff $a$ is red. (Its argument-structures are structures $\langle A, a \rangle$, where A is a universe and $a \in A$.)

(b) The first-order identity relation is represented by an operator, =, which for every universe A is assigned a function, $=_A: A \times A \rightarrow \{T, F\}$, such that for any $a, b \in A$, $=_A(a, b) = T$ iff $a = b$. (Its argument-structures are structures $\langle A, a, b \rangle$ where $a, b \in A$.)

(c) The first-order existential quantifier is represented by an operator, $\exists$, such that $\exists_A: P(A) \rightarrow \{T, F\}$, and for every $B \subseteq A$: $\exists_A(B) = T$ iff B is not empty.[5] (Its argument-structures are structures $\langle A, B \rangle$ where $B \subseteq A$.)

(d) The first-order monadic cardinality quantifiers, "There are exactly $\kappa$ things such that," where $\kappa$ is any cardinal, finite or infinite, are represented by operators, K, of the same kind as $\exists$, and such that for every $B \subseteq A$: $K_A(B) = T$ iff the cardinality of B—$|B|$—is $\kappa$. (Their argument-structures are the same as those of $\exists$.)

(e) The first-order monadic quantifier "It is a property of humans" is represented by an operator H of the same kind as $\exists$, and such that for every $B \subseteq A$: $H_A(B) = T$

[4] Types of arguments are the same as types of operators. (See (iii) above.)
[5] $P(A)$ is the power set of A.

iff all the members of B are humans. (Its argument-structures are the same as those of $\exists$.)

(f) The first-order polyadic quantifier "Is a well-ordering" is represented by an operator W such that $W_A : P(A \times A) \to \{T, F\}$, and for every $R \subseteq A \times A$: $W_A(R) = T$ iff R well-orders A. (Its argument-structures are structures $\langle A, R\rangle$ where $R \subseteq A \times A$.)

And so on.

We now define:

*An n-place operator O is invariant under all isomorphisms of its argument-structures*
iff
for any of its argument-structures, $\langle A, \beta_1, \dots, \beta_n\rangle$ and $\langle A', \beta'_1, \dots, \beta'_n\rangle$: if $\langle A, \beta_1, \dots, \beta_n\rangle \cong \langle A', \beta'_1, \dots, \beta'_n\rangle$, then $O_A(\beta_1, \dots, \beta_n) = O_{A'}(\beta'_1, \dots, \beta'_n)$.

It is easy to see that all the standard logical operators—e.g., (b) and (c), as well as the logical connectives when considered as objectual operators[6]—are logical according to this criterion, and that all blatantly non-logical operators—operators like (a) and (e)—are not. But the Invariance-under-Isomorphism criterion is a substantive criterion that does not just repeat what we think of as logical prior to a systematic, theoretical reflection. Quantifiers like the infinitistic (d)'s and (f) are also logical. Other non-standard logical operators include the uncountability quantifier and the monadic and relational "most."[7] In general, mathematical operators as they appear in first-order theories—e.g., the first-order set-membership operator ($\in$)—are not logical, but when raised to a higher order—e.g., the second-order set-membership operator ($\in$)—they are logical.[8]

[6] e.g., the logical connective "&" when considered as an objectual operator (as when it appears in an open formula of the form "Bx and Cx") is represented by an operator $\cap^2$ such that $\cap^2{}_A : P(A) \times P(A) \to P(A)$ and for every $B, C \subseteq A$: $\cap_A(B, C)$ = the intersection of B and C. For the sake of determining its logicality we represent this functional quantifier by the relational quantifier $\cap^3$ such that $\cap^3{}_A : P(A) \times P(A) \times P(A) \to \{T, F\}$, and for any $B, C, D \subseteq A$: $\cap^3{}_A(B,C,D) = T$ iff D is the intersection of B and C. (Its argument-structures are structures $\langle A, B, C, D\rangle$ where $B, C, D \subseteq A$.)

[7] These are defined as follows:

(i) The first-order monadic quantifier "There are uncountably many" is represented by an operator, U, of the same kind as $\exists$, and such that for every $B \subseteq A$: $U_A(B) = T$ iff B is uncountable. (Its argument-structures are the same as those of $\exists$.)

(ii) The first-order monadic quantifier "Most" (as in "Most things are B") is represented by an operator, $M^1$, of the same kind as $\exists$, and such that for every $B \subseteq A$: $M^1{}_A(B) = T$ iff $|B| > |A - B|$. (Its argument-structures are the same as those of $\exists$.)

(iii) The first-order relational quantifier "Most" (as in "Most B's are C's") is represented by an operator $M^2$ such that $M^2{}_A : P(A) \times P(A) \to \{T, F\}$, and for every $B,C \subseteq A$ : $M^2{}_A(B,C) = T$ iff $|B \cap C| > |B - C|$. (Its argument-structures are structures $\langle A, B, C\rangle$ where $B,C \subseteq A$.)

[8] These operators are defined as follows:

(i) The first-order membership relation is represented by an operator, $\in$, of the same type as $=$, such that for any *a*, *b* in A, $\in_A(a, b) = T$ iff *b* is a set and *a* is a member of *b*. (Its argument-structures are the same as those of $=$.)



What about the logical connectives considered, as they usually are, as propositional operators? There are two ways to deal with propositional connectives: either we expand the notion of structure so that the Invariance-under-Isomorphism criterion applies to such operators, or we give a disjunctive criterion of logicality, dealing with propositional and objectual operators separately. Not surprisingly mathematicians (e.g., Tarski and Lindström) have opted for the former, but as a philosopher I prefer the latter. I think that the philosophical idea underlying logicality is realized on different levels of abstraction for the two types of operator, and to signal this difference I define:

**Logicality**: An operator is logical iff it either satisfies the Truth-Functionality criterion for propositional operators or it satisfies the Invariance-under-Isomorphism criterion for objectual operators.

Leaving the relation between Truth-Functionality and Invariance-under-Isomorphism aside for a moment, our next question is: What is the philosophical meaning of the Invariance-under-Isomorphism criterion?

## PHILOSOPHICAL SIGNIFICANCE OF INVARIANCE-UNDER-ISOMORPHISM

The idea that logic is characterized by an invariance condition—i.e., by the things it does not distinguish between—has a long history. Kant, for example, says that "[general logic] treats of understanding without any regard to difference in the objects to which the understanding may be directed" (1781/7: A52/B76), and Frege says that "[p]ure logic . . . disregard[s] the particular characteristics of objects" (1879: 5). But this trait can be construed in different ways, and two philosophical construals of Invariance-under-Isomorphism are: (a) generality (Tarski 1966), and (b) formality (Sher 1991).

### Generality

In proposing his logicality criterion Tarski continually emphasized the fact that notions invariant under more transformations are more general than notions invariant under fewer[9] transformations. Thus, in geometry, we have more transformations preserving the ratio of distances between points than transformations preserving the actual distances between them, and more transformations preserving openness than transformations preserving the ratio of distances. Accordingly, notions invariant

(ii) The second-order membership relation is represented by an operator, $\in$, which for every universe A is assigned a function, $\in_A$: $A \times P(A) \to \{T, F\}$, such that for any *a* in A and B included in or equal to A, $\in_A(a, B) = T$ iff *a* is a member of B. (Its argument-structures are structures $\langle A, a, B\rangle$ where *a* is in A and B is included in or is equal to A.)

[9] "Fewer" here means "proper subset" rather than "smaller cardinality," as when we say that the set of odd positive integers has fewer elements than the set of positive integers.

under transformations preserving openness are more general than those invariant under transformations preserving the ratio of distances, and the latter are more general than notions invariant under transformations preserving actual distances.

To obtain *the most general* notions we renounce all restrictive conditions on the transformations partaking in the invariance condition. And invariance under all (bisective) transformations characterizes the *logical notions*. The distinctive mark of logicality, on this conception, is thus *utmost generality*, and this trait is captured by the Invariance-under-Isomorphism (or permutation) criterion.

Thus Tarski says:

> Now suppose we continue this idea, and consider still wider classes of transformations. In the *extreme case*, we would consider the class of *all* one-one transformations of the space, or universe of discourse, or 'world', onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have very few notions, all of a very general character. I suggest that they are the logical notions.
>
> (1966: 149; my underline)

It is natural to associate *utmost generality* with another characteristic feature of logic, *topic neutrality*, and this seems to strengthen the plausibility of interpreting Invariance-under Isomorphism as maximal generality.

But does Invariance-under-Isomorphism yield *the* most general notions? In "logicality and Invariance" (2006) Denis Bonnay challenges the identification of Invariance-under-Isomorphism with maximal generality:

> The [interpretation] in terms of generality rests on the assumption that invariance under the biggest class of transformations yields maximal generality. The idea is that the group of all permutations is as "big" as one might wish, because in that case the transformations do not respect any extra-structure, such as *e.g.*, the topological structure of the space. Let us have a closer look at this idea. Permutation invariance just says that as soon as there is an automorphism linking ⟨M, A⟩ and ⟨M, A′⟩, a quantifier Q acting on M has to give A and A′ the same value. On the one hand, this is indeed liberal, because no further structure beyond the extensions A and A′ on M is taken into account. But on the other hand, this is quite demanding: for ⟨M, A⟩ and ⟨M, A′⟩ to be similar from a logical point of view, they have to share exactly the same structure—they have to be isomorphic. Now there are a lot of other concepts of similarity between structures which are used in model theory and in algebra which are far less demanding. Instead of requiring the structure to be fully preserved, they lower the requirement to some kind of approximate preservation. Why should we refrain from resorting to these other concepts? To sum up, even if one grants that generality is a good way to approach logicality, there is no evidence that the class of all permutations is the best applicant for the job.
>
> (Bonnay 2008: 38)

Bonnay's point is well taken. In the extreme case we can remove all constraints on the functions involved, requiring logical operators to be invariant under *all functions* (from argument-structures to argument-structures of a given kind) *whatsoever*. This would give us the utmost general notions (in one reasonable sense of the word), but these notions would have very little to do with what we think of as logic. All the standard logical notions would fail this criterion, and the notions that would satisfy it would be such notions as: "is an individual," "is a property of individuals," "is an n-place relation of individuals (n > 1)," "is a property of properties of individuals," etc. Logic, according to this characterization, would be a theory of *semantic types*, not a theory of *inference* (or *transmission of truth*) as we intend it to be. I conclude that: (a) Invariance-under-Isomorphism does not mean utmost generality, and (b) if we want to preserve any semblance to what we intuitively mean by logic, we cannot regard utmost generality, or for that matter topic neutrality, as *the* mark of logic.



## Formality

On my interpretation (Sher 1991 and elsewhere), the Invariance-under-Isomorphism criterion is a criterion of *formality* or *structurality*: isomorphic structures are formally identical; identity-up-to-isomorphism is formal identity. The basic idea is that logic is a theory of reasoning based on the formal (structural) laws governing our thinking on the one hand and reality on the other, and the Invariance-under-Isomorphism criterion says that to be formal is to treat isomorphic structures as the same structures. Formal operators do not distinguish between isomorphic arguments (or rather between isomorphic argument-structures, since some formal features of arguments depend on the formal traits of the underlying universe).

The view that Invariance-under-Isomorphism captures the concept of formality (or structurality) is well-known from the philosophy of mathematics. Structuralists, in particular, view mathematics as the science of structure (or formal structure), and Invariance-under-Isomorphism as a mark of structurality. The Invariance-under-Isomorphism criterion characterizes logic as a theory of formal or structural inference, inference based on the laws governing formal or structural operators.

What is the relation between logic and mathematics under this interpretation? I will attend to this question in the next section, but in the meantime let me say that on the "formalist" conception of logic, logic and mathematics are interconnected theories, approaching the same topic, *the formal*, from different, yet interrelated, perspectives. Mathematics investigates the laws of formal structure; logic applies these laws in general reasoning. Logic includes mathematics, raised to a higher-order, so it can be applied in inference in general. The idea is that formal operators—union, intersection, complementation, non-emptiness, majority ("most"), finiteness, and others—are applicable to structures of objects studied in all areas of knowledge, and therefore inferences based on the laws governing them are valid in all areas.

This universal applicability of the formal operators explains logic's generality and topic neutrality. Logic does not distinguish between different topics of discourse since the formal laws governing the behavior of individuals, properties, and relations in different areas are the same. (In all areas individuals are identical to themselves, the union of non-empty properties is non-empty, etc.) Their differences concern something other than these formal laws, and logic abstracts from such differences. Comparing the two characterizations of logic associated with the Invariance-under-Isomorphism criterion, then, we can say that the *formality* of logic *ensures* its *generality* (not absolute generality, but a very high degree of generality), while the *generality* of logic *does not ensure* its *formality*. This is but one advantage of taking formality rather than generality as the mark of logic. In the remainder of this chapter I will assume Invariance-under-Isomorphism characterizes logicality as formality.

## PHILOSOPHICAL JUSTIFICATION OF THE INVARIANCE-UNDER-ISOMORPHISM CRITERION

Now that we have a basic understanding of the Invariance-under-Isomorphism criterion, our next task is to provide a philosophical justification for this criterion. I think it is quite clear that this criterion satisfies the first methodological desideratum mentioned above, namely, systematicity and informativeness (i.e., a genuine principle of logicality as opposed to a definition by enumeration).[10] But it also satisfies the other desiderata. For example, it has opened new areas of research in mathematics and linguistics and helped solve standing problems in both disciplines.[11] Here, however, I would like to focus on substantive philosophical points that support this criterion, i.e., give it what may be called "a foundational justification." By this I mean showing how the philosophical conception of logic associated with this criterion—namely, the "formalist" conception briefly delineated in the last section—is capable of providing a foundation for logic largely due to its association with this criterion.

### Methodological quandary: holistic vs. foundationalist foundation

In thinking about a foundation for logic most of us think in foundationalist terms: we think that the only way to establish logic is by using epistemic resources that are more basic than those produced by logic itself. And this leads us to a pessimistic conclusion: since no sufficiently rich branch of knowledge is more basic than logic, there is no possibility of establishing logic; a foundation for, or a justification of, logic is in principle impossible. The source of the problem, it is easy to see, is the foundationalist conception of the foundation (justification, grounding) relation as intuitively *strongly ordered*. In the ideal case, foundationalism requires that our entire system of knowledge be ordered by an anti-reflexive partial-ordering, that this ordering have an absolute base consisting of minimal (initial, atomic) elements, and that each non-minimal element

[10] Among other things, the Invariance-under-Isomorphism criterion does for objectual logical operators what the Boolean, truth-functional criterion did for propositional logical operators, namely, provide a complete, precise, systematic definition, fleshing out their structure, and explaining how they "work." (How they work is best shown by a "constructive" or "bottom-top" definition of logical operators. Such a definition is formulated in Sher 1991, Ch. 4, and is informally described in Sher 1996*b*.)

[11] For example, it has led to the development of "model-theoretic logic" and "generalized-quantifier theory." Some remarkable results of these new fields are Lindström's characterization of (standard) first-order logic, Keisler's completeness proof for first-order logic with the quantifier "uncountably many," the solution to the problem of determiners in linguistic semantics, and the theories of polyadic and branching quantifiers in natural language. The literature here is enormous. For a small sampling see Keisler (1970), Lindström (1974), Barwise and Feferman (1985), Higginbotham and May (1981), Barwise and Cooper (1981), Keenan and Stavi (1986), Van Benthem (1983 and 1989), Westerståhl (1985 and 1987), Keenan (1987), and Sher (1991, chs. 2, 4, and 5).



in the system be connected to each minimal element grounding it by a finite chain. This central feature of the foundationalist method is its Achilles heel; due to it, foundationalism has, in principle, no resources for grounding the basic constituents of knowledge—the disciplines constituting the lowest echelon in the foundationalist hierarchy. In particular, foundationalism is incapable of providing a foundation for logic. As a basic branch of knowledge, logic can partake in the foundation of other sciences, but no science (or combination of sciences) can provide a foundation for logic. Having postulated (i) that any resource for founding logic must be more basic than the resources produced by logic itself, and (ii) that there are no (or not enough) resources more basic than those produced by logic, foundationalism is incapable of founding logic.

In view of these considerations, it is clear that a foundation for logic must be *holistic*. I will not be able to explain in great detail the idea of a holistic foundation, or *foundational holism*, here. (For an extended discussion see Sher 2006.) But a few points have to be made:

- Foundational holism would provide a foundation for logic in the sense of describing its basic mechanisms, justifying the definitions of central meta-logical concepts, solving standing problems in the philosophy of logic, identifying constraints on logic, elucidating the relation between logicality and related concepts, sorting out and accounting for the distinctive characteristics of logic, explaining logic's role in our system of knowledge, throwing light on the relation between logic and mathematics, providing critical tools for detecting errors and making improvements in logical theory, etc.
- Foundational holism is not coherentist. It requires that knowledge be grounded in reality; in fact, it strengthens foundationalism by requiring that every branch of knowledge be grounded in reality. But, being holistic, it permits us to use all the resources available to us in providing such a grounding.
- Foundational holism does not require an absolute, infallible foundation, but it requires a solid foundation.
- Foundational holism requires a theoretical, and not just an intuitive, grounding of logic.
- Foundational holism rejects vicious circularity, but not circularity per se. Which circularity is vicious is determined by holistic methods.

Having made these methodological points, I will proceed to show how the formalist conception of logic accomplishes some of the foundationalist tasks mentioned above due to its association with the Invariance-under-Isomorphism criterion.

A. *Explanation of logic's connection to truth*. It is commonplace to say that as a theory of logical truth (truth of a certain kind) and of logical consequence (transmission of truth of a certain kind) logic is intimately connected with truth. But what, exactly, are the nature of this connection and its constraints on logic? Let us start with general theoretical considerations.

Assuming the classical idea that truth importantly involves some correspondence relation between truth-bearers and reality, let us consider two truth-bearers, $S_1$ and

$S_2$, whose truth-conditions straightforwardly and paradigmatically exemplify this idea. (For the sake of simplicity, let us further assume that $S_1$ and $S_2$ are distinct and non-synonymous.) Now, suppose that according to some logical theory, L, $S_2$ is a logical consequence of $S_1$. In symbols:

(1) (*Level of Logic*) $S_1 \models^{12} S_2$.

Further suppose that $S_1$ is true. Then (1) says that the truth of $S_1$ extends to, or is transmitted to, or is preserved by, $S_2$:

(2) (*Level of Language*) $T(S_1) \longrightarrow T(S_2)$.

((1) says something stronger than that, but let us attend to the weaker claim first.)

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the situations that have to be realized for $S_1$ and $S_2$ to be true and that would guarantee their truth were they to be realized. Figuratively:

(3) (*Level of Language*) $T(S_1)$ $T(S_2)$.
$\Updownarrow$ $\Updownarrow$
(*Level of World*) $\mathfrak{S}_1$ $\mathfrak{S}_2$.

Now, suppose that in the world $\mathfrak{S}_1$ is the case but $\mathfrak{S}_2$ is not. (In the extreme case, $\mathfrak{S}_1$ rules out $\mathfrak{S}_2$.) I.e.,

(4) (*Level of World*) $\langle \mathfrak{S}_1, \text{not } \mathfrak{S}_2 \rangle$(in the extreme case : $\mathfrak{S}_1 \Rightarrow \text{not } \mathfrak{S}_2$).

Then, our logical theory is wrong. No matter what L says, $S_2$ is *not* a logical consequence of $S_1$:

(5) (*Level of Logic*) $S_1 \not\models S_2$.

Logic, indeed, is constrained by truth more deeply than the above consideration suggests. Suppose that in the world both $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the case, but $\mathfrak{S}_1$ being the case does not require $\mathfrak{S}_2$ being the case:

(6) (*Level of World*) $\langle \mathfrak{S}_1, \mathfrak{S}_2, \quad \mathfrak{S}_1 \not\Rightarrow \mathfrak{S}_2 \rangle$.

Then, again:

(7) (*Level of Language*) $S_1 \not\models S_2$.

Now, an adequate criterion of logicality has to explain, or be incorporated in an account that explains, this alethic constraint on logic. The formalist interpretation of the Invariance-under-Isomorphism criterion delineated in the last section is embedded in a "formalist" account of logic that does just that. We can sum up its main points as follows:[13]

(i) The logical constituents of truth-bearers—especially, their *logical constants*—represent formal properties, relations, and functions, where formality is interpreted as Invariance-under-Isomorphism.

[12] '$\models$' is a symbol of an unspecified kind for logical consequence. "$S_1 \models S_2$" reads: "$S_1$ logically implies $S_2$."

[13] For a more detailed account see Sher (1991) and related papers.



(ii) The logical form of truth-bearers is obtained by holding their logical constants fixed and treating their non-logical constants as variable.

(iii) Corresponding to a truth-bearer S is a situation, $\mathfrak{S}$, that would make S true if it were to be realized; corresponding to the logical form of S is the formal skeleton of $\mathfrak{S}$, which contains those parameters of $\mathfrak{S}$ which correspond to its logical constituents. For example, corresponding to "Something is white and round" is a structure, ⟨A, B, C⟩ where A is the intended universe of discourse, B is the collection of white things in A, C is the collection of round things in A, and the intersection of B and C is not empty. The formal skeleton of $\mathfrak{S}$ contains the formal parameters of $\mathfrak{S}$ corresponding to the logical constants of S, namely: intersection and non-emptiness (a cardinality parameter).

(iv) Logical consequence is a relation between truth-bearers which represents a universal formal law connecting the situation corresponding to the "premise" truth-bearers to the one corresponding to the "conclusion" truth-bearer. Alternatively, logical consequence correspond to, and is largely due to, a law connecting the formal skeleton of the "premise" situations to the formal skeleton of the "conclusion" situation. This law is universal in the sense that it holds in all formally possible situations, or in all possible formal-structures. For example:

*Something* is white

is a logical-consequence of

*Something* is white *and* round

because it is a formal law that whenever an intersection of two subsets is not empty, the first of these subsets is not empty; it is not a logical consequence of

*Something* is white *or* round

because it is not a formal law that whenever a union of two subsets is not empty, the first of these subsets is not empty.

If we regard formal laws as formally necessary, we can concisely represent the present conception of logical consequence thus:

*Level of Language*: $S_1$ *logically implies* $S_2$.
$\Updownarrow$
*Level of World*: $\mathfrak{S}_1$ *formally necessitates* $\mathfrak{S}_2$.

(v) In contemporary (Tarskian) semantics we represent the formally possible situations vis-à-vis a given language by the totality of *models* for that language. Universal formal laws are represented by *regularities across all models*.

(vi) This explains the standard (Tarskian) semantic definition of logical consequence: S is a logical-consequence of K iff S is true in all models (i.e., formally possible situations) in which all the members of K are true, i.e., when S is a logical consequence of K, this is due to some formal law connecting the situations corresponding to S and K.

*Note*: This account assumes a background theory of formal structure, used to formulate the logicality criterion, delineate the totality of formally possible situations represented by models, determine the laws governing them (i.e., the formal laws underlying logical consequence), etc. The appeal to such a background theory is licensed by the holistic methodology of the account. This is an important point that is easy to miss. Indeed, it is so common to associate the foundational goal with the foundationalist method and the holistic method with the renunciation of this goal, that many philosophers evaluate *any* foundational proposal based on foundationalist standards.[14]

The holistic approach enables us to maneuver the limitations of the background theory in a rational and effective manner. Consider, for example, the incompleteness phenomenon. The holist reasons that in the same way that we are forced to use incomplete mathematical background-theories in providing a foundation for physics, so we are forced to use incomplete mathematical background-theories in providing a foundation for logic. To the extent that no perfect, complete theory of formal structure is available (temporarily or in principle) but some quite advanced theories do exist, we rely on the best background theory we can find, and avow ignorance with respect to those cases of logical truth and consequence that this theory cannot handle.

Yet the holist can still hold on to the classical concept of truth, i.e., ensure that there is a fact of the matter about how, say, the "continuum quantifier" behaves. This he does by using a *complete* version of his chosen background theory—specifically, the theory of some *model* of his theory of formal structure—to determine *facts* about the behavior of logical operators and the laws governing them, and an *incomplete* axiomatization to derive whatever *knowledge* he can have of those facts. Truth is anchored in a complete (if inaccessible) theory of formal structure, knowledge—in an effectively axiomatized, hence accessible (if incomplete) version of that theory. The facts are as they are, but knowledge is in principle limited.

## Explanation of logic's role in knowledge and its place in our system of knowledge

The formalist account of logic enables us to explain the role played by logic in our system of knowledge (and to the extent that this explanation is compelling, it is also

[14] In the case of logic, one relevant example is Etchemendy (this volume). Etchemendy thinks that due to the incompleteness of any reasonable background theory of formal structure we cannot establish the formal necessity of Tarskian consequences. (See his criticism of Sher 1996*a* in fn. 24 of Chapter 11). This claim is right for a foundationalist, who requires absolute certainty and intuitive completeness (hence also technical completeness) of a putative foundation, but not for a holist who, contesting the appropriateness of such demands, allows the background theory of formal structure, like all human theories, to be short of perfect (and technically incomplete). I will presently attend to the incompleteness problem in a little more detail, but the point here is that while Etchemendy himself is, for all I know, a holist, he applies foundationalist standards to the foundational claims in Sher (1996*a*), despite the fact that they are explicitly offered as holistic claims. He seems not to appreciate the possibility of a holistic foundation, or foundational holism.



supported by it). According to this explanation logic plays a dual role in knowledge: first, it sets general constraints on what counts as knowledge, and second, it creates useful tools for expanding and correcting our knowledge. Let us consider the latter first:

### *Expansion of knowledge*

Being finite and relatively short-living creatures, we cannot hope to establish all our knowledge directly but have to resort to such indirect means as *inference* to obtain a considerable portion of our knowledge. In inference we use our knowledge of the relations between objects or situations plus some knowledge of these objects or situations to obtain new knowledge which, as inferred knowledge, does not require independent verification. For example, if we have knowledge about the chemical constitution of objects and the relations between chemical structures, we can use this knowledge to obtain new knowledge about objects. But while chemical laws enable us to expand our knowledge in a small number of areas, formal laws enable us to expand it in all areas. Given that formal features of objects are constantly referred to in all discourse—one cannot talk about anything without saying that certain objects are in the *complement* or *intersection* of certain properties, that certain properties are *non-empty*, or *universal*, or have *κ objects* falling under them, etc.—we can use our knowledge of these features to develop a wholesale method of expanding our knowledge. Logic, on this conception, utilizes our knowledge of the formal behavior of objects to formulate rules of inference that sanction our movement from what we know to what (prior to this movement) we did not know. Knowledge of some formal laws may be more useful for expanding our overall knowledge than knowledge of others, so it might be useful to build limited logical systems geared to those features. But in principle logic can provide us with rules for expanding our knowledge based on any laws governing the formal behavior of objects.

### *Constraints on knowledge*

Due to the prevalence of formal features of objects and our constant reference to such features in discourse and theorizing, the threat of formal errors in our system of knowledge looms large. But due to the fact that the formal does not distinguish between different domains of knowledge, it is possible to take care of such errors in "one fell swoop", so to speak, i.e., in a way that protects all (or most) fields of knowledge at once. This opportunity is seized by logic. Logic builds into our language rules that prevent us from making errors pertaining to the (law-governed) formal behavior of objects in any area. For example, by telling us that statements of the form "Φa & ~Φa" are false (or that a combination of statements of the form "Φa" and "~Φa" is inconsistent) logic prevents us from making certain errors concerning the behavior of objects under the complementarity operation (in any field). By telling us that inferences of the form "(∀x)(∃y)Φxy; therefore (∃y)(∀x)Φxy" are invalid, logic prevents us from assuming certain symmetries exist where they do not. And so on.

## Solution to Tarski's problem

In his 1936 chapter, "On the Concept of Logical Consequence," Tarski sought a definition of "logical consequence" that would satisfy two intuitive constraints:

> Certain considerations of an intuitive nature will form our starting-point. Consider any class *K* of sentences and a sentence *X* which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class *K* consists only of true sentences and the sentence *X* is false. Moreover, since we are concerned here with the concept of logical, i.e., *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence *X* or the sentences of the class *K* refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects. The two circumstances just indicated, . . . seem to be very characteristic and essential for the proper concept of consequence.
>
> (Tarski 1936: 414–15)

Based on these considerations Tarski formulated his semantic definition of "logical consequence":

> *The sentence X* follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*
>
> (Ibid.: 417)

Is this an adequate definition? Does it satisfy the intuitive constraints? At first Tarski gave a positive answer:

> It seems to me that everyone who understands the content of the above definition must admit that it agrees quite well with common usage. This becomes still clearer from its various consequences. In particular, it can be proved, on the basis of this definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences.
>
> (Ibid.)

But soon he qualified his answer:

> I am not at all of the opinion that in the result of the above discussion the problem of a materially adequate definition of the concept of consequence has been completely solved. On the contrary, I still see several open questions, . . . one of which—perhaps the most important—I shall point out here.
>
> (Ibid.: 418)

This question was the demarcation of logical constants:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand, no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage.
>
> (Ibid.: 418–19)



These qualifications led Tarski to conclude his chapter on a skeptical note:

> Further research will doubtless greatly clarify the problem which interests us. Perhaps it will be possible to find important objective arguments which will enable us to justify the traditional boundary between logical and extra-logical expressions. But I also consider it to be quite possible that investigations will bring no positive results in this direction, so that we shall be compelled to regard such concepts as 'logical consequence' and . . . 'tautology' as relative concepts which must, on each occasion, be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra-logical.
>
> (Ibid.: 420)

The Invariance-under-Isomorphism criterion offers a positive solution to Tarski's problem. It offers a demarcation of logical operators under which Tarski's definition of logical consequence can be shown to satisfy the intuitive constraints. To see how, consider the following:

1. Tarski set two intuitive constraints on an adequate definition of logical consequence:

   (C1) Necessity: When X follows logically from K, X follows necessarily from K.

   (C2) Formality: When X follows logically from K, X follows formally from K.

2. Regardless of what Tarski himself understood by necessity, if we show that his definition satisfies a robust standard of necessity, we will have shown that it satisfies whatever weaker standard he might have had in mind.
3. Formality can be interpreted both syntactically and semantically. Philosophers often think of formality syntactically, but the key to vindicating Tarski's definition is to think of it semantically.
4. Tarski himself offers the key to a semantic interpretation of formality:

   > [As a formal relation, logical consequence] cannot be influenced in any way by . . . knowledge of the objects to which the sentence X or the sentences of the class K refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects.
   >
   > (Ibid.: 414–15, cited above)

5. This paragraph suggests that the formal is characterized by its inability to distinguish the identity of objects in a given universe of discourse. This is an invariance characterization: formal relations are invariant under replacements of objects. Now, if we interpret "replacement" as "1–1 and onto transformation or mapping," and "replacement of objects" as "replacement of objects of all types induced by replacement of the individuals in a given universe of discourse," then we get the Invariance-under-Isomorphism criterion of logicality.

6. Under this criterion all logical operators are derivable from mathematical operators by raising them to a higher order (as we have seen on p. 307), and in this sense they are essentially mathematical.
7. But the laws governing mathematical operators are intuitively formal and necessary (where this necessity is an especially strong kind of necessity, stronger than biological, physical, and even metaphysical necessity). Therefore, if logical consequence is due to the formal (or mathematical) laws governing the logical operators, logical consequences are formal and (strongly) necessary.
8. Now, on a formalist reading Tarski's definition does satisfy the antecedent of this conditional. The totality of models represents the totality of formal possibilities; logical consequences preserve truth across all models; they do so due to the logical structure of the sentences involved; this logical structure reflects the formal skeleton of the situations described by those sentences; therefore the preservation of truth is due to connections that hold between the formal skeletons of the situations involved in all formal possibilities; and formal connections persisting through the totality of formal possibilities are laws of formal structure. It follows that consequences satisfying Tarski's definition are formal and necessary, as required by the intuitive constraints (however strong the necessity constraint is taken to be).[15]

## Explanation of the distinctive characteristics of logic

Logic is often characterized by its basicness, generality, topic-neutrality, necessity, formality, strong normative force, certainty, a-priority, and/or analyticity. While, as foundational holists, we reject the purported analyticity of logic and qualify its a-priority, we can explain its other characteristics (including quasi-apriority) based on the Invariance-under-Isomorphism criterion, i.e., explain why the laws of logic and its consequences are as basic, general, topic-neutral, formal, strongly normative, and highly certain as they appear to us to be, and to what degree they are a-priori.

We have already seen how the Invariance-under-Isomorphism criterion, either alone or together with other elements of the formalist account, explains the *formality*,

[15] In defending the adequacy of Tarski's definition it may seem that we have to confront Etchemendy's 1990 challenge to it, but in fact we don't. Etchemendy considers two conceptions of logic: the so-called representational and interpretational conceptions. But the formalist conception of logic offered here (and in Sher 1991) falls under neither category. Since Etchemendy's criticisms center on features of those conceptions that are not shared by the present conception, his criticisms do not concern us here. This includes his claim that the problem of logical constants is a "red herring." Etchemendy regards the problem of logical constants as a red herring not because he thinks logical constants do not pose a genuine problem to Tarski's definition, but because he thinks that Tarski's definition is plagued by other problems as well and merely solving the logical constants problem will not by itself establish its adequacy. However, the additional problems Etchemendy alludes to are specific to the interpretational construal of logic and do not arise on the formalist construal. Therefore, on that construal the problem of logical constants, far from being a "red herring," is the main obstacle to the adequacy of Tarski's definition. For a fuller critique of Etchemendy's (1990) see Sher (1996*a*).



*generality*, *topic-neutrality*, and *necessity* of logic. Let us, then, turn to the other characteristics.

*Basicness* and *strong normative force*. Logic is intuitively more basic than other disciplines. The grounding of geography, biology, and chemistry involves establishing their *logical* consistency, i.e., establishing that their laws obey the laws of logic, but the grounding of logic does not involve establishing that its laws obey the laws of geography, biology, and chemistry. This gap is related to a gap in the normative force of logic and other disciplines. Chemistry, biology, and geography have to attend to the strictures of logic, but logic need not attend to their strictures. Logic has normative authority over these disciplines, but not vice versa. The Invariance-under-Isomorphism criterion explains why this is so: Since chemical properties are not preserved under isomorphisms, logic has a stronger invariance property than chemistry. As a result, logic does not distinguish chemical differences between objects and is not subject to the laws governing chemical properties. But chemistry does distinguish formal differences between objects; for example, it distinguishes between *one* atom and *two* atoms. So chemistry is subject to the laws of formal structure. For example, chemistry is bound by the law

$$(\exists!2x)\Phi \supset \sim(\exists!3x)\Phi,$$

as in

$(\exists!2x)$ x *is a Hydrogen atom in water molecule w* $\supset \sim(\exists!3x)$ x *is a Hydrogen atom in w.*

And the same holds for most other disciplines. (The case of mathematics will be discussed separately below.)

*Certainty and quasi-a-priority*. Logic has a relatively high degree of certainty, not in the sense that we are less likely to make errors in applying the logical laws than other laws, but in the sense that the logical laws themselves are unlikely to be refuted by our empirical discoveries. The Invariance-under-Isomorphism criterion explains why logic is immune to refutation in this sense. Since most of our empirical discoveries do not concern the formal regularities in the behavior of objects—i.e., regularities governing features of objects that are invariant under isomorphism—logic is not affected by most of these discoveries, and in this sense it is resistant to refutation and, furthermore, *a-priori-like*. Now, if formal laws were completely immune to discoveries having any empirical element, then logic would be strictly a-priori. But holism allows a certain degree of interconnection between all disciplines, hence on the holistic approach logic is only quasi-a-priori. What kind of empirical discoveries could affect logic? Empirical discoveries affect logic only in very rare cases, and therefore we have no ready examples, but one challenge to classical logic did come from physics (Birkhoff and von Neumann 1936), and by extrapolating from it we could arrive at a possible scenario in which empirical discoveries would affect logic. Suppose we discover that in some region of reality (e.g., the quantum region) objects or states behave in a way that is radically different from what we have observed elsewhere, and we have good reasons to believe that it concerns the basic formal behavior of objects (states, properties). For example, suppose we have good reasons to believe that their behavior is deeply non-Boolean. Then this discovery would pose serious questions to classical logic.

## Explanation of the relation between logic and mathematics

Ever since Frege, logic and mathematics have been treated as closely related disciplines whose relation requires an explanation. And one of the least noted, but methodologically most important achievements, of Frege's logicism was the enormous economy it brought to the philosophical tasks of explaining the nature of logic and mathematics and providing them with a foundation. By reducing mathematics to logic, logicism reduced two mysteries to one. Instead of having to explain both the nature of logic and the nature of mathematics we now had to explain only the nature of logic; and instead of the monumental task of constructing both a foundation for logic and a foundation for mathematics, we had the more manageable task of constructing a foundation only for logic. However, the search for a foundation for logic (independently of mathematics) led to nowhere. The most influential attempt to construct an account of logic that would complement logicism—Carnap's conventionalism—has by and large been discarded, and this, together with the almost unanimous rejection of logicism itself, has left us, once again, with the extremely difficult task of providing an explanatory account and a foundation both for logic and for mathematics.

The formalist account of logic, with its Invariance-under-Isomorphism criterion of logicality, offers an explanation of the relation between logic and mathematics that has the same methodological advantage as Frege's explanation without having its shortcomings. Like Frege's account, it reduces the two fields to one, hence the two foundational tasks to one. But this time it is logic that is reduced to mathematics rather than mathematics to logic. Or, alternatively, both logic and mathematics are reduced to the formal. Mathematics, in this account, builds a theory of formal structure, and logic provides a method of inference based on this theory. I will call the new approach "mathematicism." If *logicism* is the view that mathematics has a logical foundation, *mathematicism* is the view that logic has a mathematical foundation. But there is a considerable methodological advantage to mathematicism over logicism. While today we have no promising foundational account of logic not centered on mathematics, we do have a number of promising foundational accounts of mathematics not centered on logic; for example, the Platonist account, the naturalist account, and the structuralist account. It is true that these accounts assume logic in the background, but since mathematicism seeks to give a *holistic* foundation for logic, this does not pose a special difficulty. Logic does not stand at the center of any of these accounts, therefore the circularity involved is (at least prima facie) not vicious.

But the current situation is even more felicitous. Not only are several accounts of mathematics compatible with logical formalism, one of these accounts, the *structuralist* account, is very close to it in spirit. This is reflected in the fact that mathematical structuralism and logical formalism share the same identity criterion: invariance under isomorphism. Invariance under isomorphism is the identity criterion of logical operators according to logical formalism, and it is also the identity criterion, or at least an identity-criterion of choice, of mathematical structures according to mathematical structuralism. Thus, Shapiro says:



> No matter how it is to be articulated, structuralism depends on a notion of two systems that exemplify the "same" structure. That is its point. . . . [W]e . . . need to articulate a relation among systems that amounts to "have the same structure".
>
> There are several relations that will do for this. . . . The first is *isomorphism*, a common (and respectable) mathematical notion. . . . Informally, it is sometimes said that isomorphism "preserves structure".
>
> (Shapiro 1997: 90–1; my underline)
>
> A purported implicit definition characterizes at most one structure if it is *categorical*—if any two models of it are isomorphic to each other.
>
> (Ibid.: 13; my underline)
>
> Because isomorphism . . . [is an] equivalence relation . . . one can informally take a structure to be an isomorphism type.
>
> (Ibid.: 92; my underline)

Indeed, it would be just as appropriate to call our account of logic "logical structuralism" as to call it "logical formalism" (and to call the structuralist account of mathematics "mathematical formalism" as to call it "mathematical structuralism").[16]

Furthermore, we can achieve the same methodological goal without reducing either discipline to the other, namely, by tracing both mathematics and logic to the same root, i.e., ***the formal (structural)***. Analytically, logic and mathematics develop in tandem from a basic engagement with the formal (the structural). We can represent their joint development along something like the following lines: In stage 1, we develop a rudimentary logic-mathematics which studies some very basic formal operators, say complementation, union, intersection, and inclusion. Based on this knowledge we develop, in stage 2, a logical framework for theories in general, and using it we develop a more sophisticated mathematical theory of formal structure (say, naive set theory). Based on this theory we develop, in stage 3, a more sophisticated logical framework, say the logical framework of standard first-order logic with its standard logical operators ($\exists$, $\forall$, $=$, and the truth-functional connectives). And using this framework we develop, in stage 4, a more advanced mathematical theory of formal structure (say, axiomatic set theory). In stage 5 we use this advanced theory to develop a criterion of logicality (for example, the Invariance-under-Isomorphism criterion) and a semantic definition of logical consequence (for

[16] My misgivings about "structuralism" is that there are many kinds of structure, not all mathematical or logical (for example, physical or biological structures which are not preserved under isomorphisms). To distinguish mathematical and logical structures from other structures I call them "formal." But "formalism" has unwanted connotations of its own, namely, Hilbertian formalism. Once we make clear, however, that our use of "formal" is semantic, this association should dissolve. In Sher (2001) I used "formal-structural" for the formalist account of logic so as to signal both its affinity to the structuralist account of mathematics and its difference from Hilbert's formalism.

example, Tarski's model-theoretic definition), and based on these, an expanded logical framework—say, so-called generalized first-order logic (or standard second-order logic). And this process may continue: using this enriched logic we may arrive at a still more powerful mathematics and, based on it, perhaps a stronger logic. And so on.

To deal with the formal in logic and in mathematics we operate on different levels. In mathematics we construe the formal as (for the most part) lower-order, in logic we construe it as (for the most part) higher-order. Take, for example, the notion of number or the notions of union, intersection, and complementation. In axiomatic arithmetic numbers are individuals, but the numerical quantifiers are operators on properties; in axiomatic set-theory union, intersection, and complementation are operations on individuals, but in logic they are operators on properties (or propositions). As studied in mathematics, these notions do not satisfy the Invariance-under-Isomorphism criterion, but as studied in logic they do. And the same holds for other formal notions: for example, the membership relation of axiomatic set theory is not logical, but the membership relation of higher-order logic is. (A more nuanced version of this account would say that the mathematician treats some mathematical concepts as non-logical and others as logical. The number theorist, for example, treats numbers as non-logical entities, but the background mathematical concepts he uses to talk, and formulate questions, about numbers—e.g., the concept of set-membership—as logical.)

Tarski's take on the philosophical ramifications of the new logicality criterion for the relation between logic and mathematics is different from mine:

> The question is often asked whether mathematics is a part of logic. Here we are interested in only one aspect of this problem, whether mathematical notions are logical notions, and not, for example, in whether mathematical truths are logical truths, which is outside our domain of discussion. Since it is now well known that the whole of mathematics can be constructed within set theory, or the theory of classes, the problem reduces to the following one: Are set-theoretical notions logical notions or not? Again, since it is known that all usual set-theoretical notions can be defined in terms of one, the notion of belonging, or the membership relation, the final form of our question is whether the membership relation is a logical one in the sense of my suggestion. The answer will seem disappointing. For we can develop set theory, the theory of the membership relation, in such a way that the answer to this question is affirmative, or we can proceed in such a say that the answer is negative. So the answer is: 'As you wish!'.
>
> (Tarski 1966: 151–2)

In my view, the new logicality criterion leads to a more intricate and interesting answer to this question. It suggests that there is a division of labor between logic and mathematics, one that leads to different practices in the two disciplines. Logic and mathematics approach the formal from two different, though complementary, perspectives, and therein lie both their similarities and their differences. Mathematics seeks to discover formal laws, logic seeks to implement them; mathematics is interested in the formal as it concerns objects, logic is interested in the formal as it concerns thought or language. And our cognitive capacities are such that discovery is best systematized in terms of individuals and their properties, implementation—in



terms of properties and relations and especially in terms of properties and relations of properties and relations. The formal is differently represented in logic and in mathematics, but at bottom it is the same in both. (For additional points and a slightly different perspective, see Sher 1991, chs. 3 and 6.)

## Tools for justifying logic's claims and detecting its errors

By using mathematical truth as a basis for logical truth, we are licensed to use mathematics, and indirectly, the tools used to justify it and detect its errors, as a tool for justifying and detecting errors in logic. For example, to the extent that mathematical or rational intuition is a tool for justifying mathematical assertions, it is also a tool for justifying the supervening logical assertions. Or to the extent that sometimes (if rarely) physical discoveries have formal ramifications, they can be used to corroborate or throw doubt on logical assertions. Or to the extent that a new claim, or an old conjecture, is proved in mathematics, we can use it to justify a logical rule of proof or a logical inference. For example, the newly discovered proof of Fermat's Last Theorem justifies all the hitherto unjustified logical rules of inference of the form:

$$(\exists! k^n x)\Phi, (\exists! l^n x)\Psi, (\forall x)(\Phi \equiv \sim\Psi); \text{ therefore } \sim(\exists! m^n x)(\Phi \vee \Psi),$$

where $n > 2$ and $k, l, m > 0$. Or if we find compelling reasons for including the Continuum Hypothesis or its negation in our theory of formal structure, we can use them to justify either the logical inference

$$(\exists! 2^{\aleph_0} x)\Phi; \text{ therefore } (\exists! \aleph_1 x)\Phi$$

or the logical inference

$$(\exists! 2^{\aleph_0} x)\Phi; \text{ therefore } \sim(\exists! \aleph_1 x)\Phi.$$

And so on.

These are some of the foundational advantages of the Invariance-under-Isomorphism criterion and the formalist theory of logic within which it is offered.

It should be noted that the Invariance-under-Isomorphism criterion also contributes to a ***critical approach to the philosophy of logic***. The prevalent philosophy of logic today adheres to the so-called "***first-order thesis***"[17] which says that standard first-order logic is the whole of logic. Very few systematic or theoretical grounds have been adduced in support of this thesis, and for the most part it has been accepted without serious argument. The Invariance-under-Isomorphism criterion challenges this thesis on several grounds. For one thing, it challenges one of the few theoretical arguments used to support it, namely, the argument from completeness (Quine 1970). Investigations connected with this criterion (e.g., Keisler 1970) have proved that standard first-order logic is definitely not the strongest (extensional) logic which

[17] This epithet is due to Barwise (1985) who made similar points to those I am about to make.

has the virtue of being complete; stronger first-order logics—for example, first-order logic with the added logical quantifier "there are uncountably many"—are also complete. More importantly, the Invariance-under-Isomorphism criterion demonstrates that a systematic, theoretical, philosophically anchored, highly explanatory, mathematically rich, and linguistically fruitful criterion of logicality is possible. In so doing it sets a new, higher standard of justification for theses concerning the scope of logic, a standard that, as far as I can judge, has not been met by any of the known justifications of the first-order thesis.

Our final task before turning to Feferman's criticisms is to show how the Invariance-under-Isomorphism criterion for objectual logical operators relates to the Boolean, truth-functional criterion for propositional logical operators (logical connectives).

*Invariance-under-Isomorphism and Truth-Functionality.* In making statements we usually work with two types of structures—objectual structures and propositional structures, and we use two types of operators—objectual operators and propositional operators. Thus, in making a statement of the form

$$\sim(\exists x)(Bx \;\&\; \sim Cx)$$

we first consider an objectual structure with two properties, B and C; then, working with the objectual operator $\sim$ (the objectual correlate of the propositional operator $\sim$, namely, complementation), we focus our attention on B and the complement of C; next, working with the objectual operator & ($\cap$) we shift our attention to the intersection of B and the complement of C; then, working with the objectual operator $\exists$, we consider the possibility that this intersection is not empty; and finally, thinking in propositional terms and using the propositional operator $\sim$, we say that this possibility is not realized: nothing is both a B and a non-C.

Now, if we commonly use operators of two types, objectual and propositional, each defined in terms of the corresponding structure, then we need two (albeit coordinated) criteria of logicality, each formulated in terms of the relevant structure.[18] Invariance-under-Isomorphism is a criterion of logicality for objectual operators, and Truth-Functionality is a criterion of logicality for propositional operators. How are they connected? The formalist answer is that the same idea—*formality*—lies at the bottom of both criteria, and the same technical device—*invariance under "isomorphism"*—is used in both, but with respect to different structures:

(I) *An objectual operator is logical iff it is invariant under all isomorphisms of its argument-structures, which are objectual.*

(II) *A propositional operator is logical iff it is invariant under all isomorphisms* of its argument-structures, which are propositional.*

[18] And we also need two related alethic predicates, "satisfaction" and "truth"—the former applying to open formulas whose operators, if any, are all objectual, and whose definition accordingly refers to objectual structures; the latter applying to closed formulas (sentences) whose new operators (i.e., those added to the operators of their open sub-formulas), if any, are all propositional, and whose definition accordingly refers to propositional structures.



What is an isomorphism* of propositional structures? When are two propositional structures formally the same? Well, formality in the domain of propositions is, on the classical approach (tentatively adopted here) *preservation of Boolean structure.* And the Boolean features of propositional structures are a generalization of the Boolean features of objectual structures. The basic parameter in this generalization is *binary structure* or *complementarity,* which is common to both objectual and propositional structures, and we can arrive from the objectual form of this parameter to its propositional form in three steps that, in the simplest case, can be described as follows:

(i) Objectual step:

Given an object *a* in a universe A and a set of objects or a property B in A, there are exactly two possibilities with respect to *a*, exactly one of which is realized: *a* is a B, *a* is a $\overline{B}$ (complement of B in A), the latter being equivalent to: *a* is not a B (in A).

(ii) Situational step:

Given the situation *s* in which *a* is a B (in A), there are exactly two possibilities with respect to *s*, exactly one of which is realized: *s* is the case, *s* is not the case (not-being-the-case being the complement of being-the-case).

(iii) Propositional step:

Given a proposition *p* corresponding to *s*, there are exactly two possibilities with respect to *p*, exactly one of which is realized: *p* is true, *p* is false (false being the complement of true).

These steps connect objectual structures to propositional structures and form a bond between the logicality criterion of objectual operators and the logicality criterion of propositional operators: an operator, objectual or propositional, is logical iff it does not distinguish the non-formal features of its argument-structures. Since the generalization from objectual to propositional structures is such that the only formal feature of a proposition is its binary value (truth or falsity), a propositional operator is logical iff it is invariance under $1-1$ mappings of propositions which transfer each proposition into a proposition with the same binary value (i.e., its truth value).

Technically, we can define:

(a) A *propositional structure* is as an $n+1$-tuple $\langle P, p_1, \ldots, p_n \rangle$, where P is the set of all propositions of a given language and $p_1, \ldots p_n$ are elements of P.

(b) An *argument-structure* for a k-place propositional operator is a propositional structure of length $k+1$.

(c) Two propositional structures $\langle P, p_1, \ldots, p_n \rangle$ and $\langle P, p'_1, \ldots, p'_m \rangle$ are isomorphic* iff $n = m$ and there is a truth-bijection from P to P, i.e., a $1-1$ truth-preserving function *f* from P onto P such that for every $1 \leq i \leq n$, $p'_i$ is the image of $p_i$ under *f*.

Truth-functionality is thus (classical) formality on the propositional level.[19]

We are now ready to consider Feferman's criticisms.

## FEFERMAN'S CRITICISMS

The Invariance-under-Isomorphism criterion is a substantive criterion, and as such it invites substantive criticisms. In "Logic, Logics and Logicism" (1999), Solomon Feferman offers three substantive criticisms of the claim that this criterion is a necessary and sufficient criterion of logicality (referred to as "the Tarski–Sher thesis"). Feferman formulates the criterion (in terms sanctioned by a certain definability result due to McGee 1996) as follows:

*An operation O across domains is a logical operation according to the Tarski–Sher thesis if and only if for each cardinal $\kappa \neq 0$ there is a formula $\phi_\kappa$ of $L_{\infty,\infty}$ which describes the action of O on domains of cardinality $\kappa$.*

(Feferman 1999: 37)

Here, however, I will continue to employ our earlier terminology in discussing his criticisms.

Feferman criticizes the Tarski–Sher thesis on three counts:

1. "*The thesis assimilates logic to mathematics, more specifically to set theory*" (ibid., my italics).

Elaboration:

The first [point], I think, speaks for itself, . . . but it will evidently depend on one's gut feelings about the nature of logic as to whether this is considered reasonable or not. For Sher, to take one example, this is no problem. Indeed, she avers that "the bounds of logic, on my view, are the bounds of mathematical reasoning. Any higher-order mathematical predicate or relation can function as a logical term, provided it is introduced in the right way into the syntactic-semantic apparatus of first order logic." ([Sher 1991], pp. xii–xiii) What that "right way" is for her is spelled out in a series of syntactic/semantic conditions . . . ([*ibid.*], pp. 54–5) . . . .[Although these conditions restrict us to] logical operation[s] . . . of type-level at most 2 . . . [this] is not set-theoretically restrictive. . . . In particular, we can express the Continuum Hypothesis and many other substantial mathematical propositions as logically determinate statements on the Tarski–Sher thesis. . . . But in so far as . . . the thesis requires the existence of set theoretical entities of a special kind, or at least of their determinate properties, it is evident that we have thereby transcended logic as the arena of universal notions independent of "what there is".

(Feferman 1999: 37–8)

2. "*The set-theoretical notions involved in explaining the semantics [of the background language] are not robust.*" (ibid.: 37; my italics)

[19] There are of course more familiar ways to construe isomorphism*; for example, using structures whose distinguished elements are truth-values. But I was looking for a construal that would be philosophically transparent, regardless of its familiarity or elegance.



Elaboration:

Point 2 is in a way subsidiary to point 1. The notion of "robustness" for set-theoretical concepts is vague, but the idea is that if logical notions are at all to be explicated set-theoretically, they should have the same meaning independent of the exact extent of the set-theoretical universe. For example, they should give equivalent results in the constructible sets and in forcing-generic extensions. Gödel's well-known concept of absoluteness provides a necessary criterion for such notions and, when applied to [the kind of operators considered by the Tarski–Sher thesis] considerably restricts those that meet this test. For example, the quantifier "there exist uncountably many $x$" would not be logical according to this restriction, since the property of being countable is not absolute.

(Ibid.: 38)

Feferman, however, qualifies his support of the absoluteness criterion somewhat:

One should be aware that the notion of absoluteness is itself relative and is sensitive to a background set theory, hence again to the question of what entities exist.

(Ibid.)

3. "*No natural explanation is given by [the Tarski–Sher Thesis] of what constitutes the* same *logical operation over arbitrary basic domains.*" (Ibid.: 37; my italics)

Elaboration:

It seems to me there is a sense in which the usual operations of the first-order predicate calculus have the *same meaning* independent of the domain of individuals over which they are applied. This characteristic is *not* captured by invariance under bijections. As McGee puts it "The Tarski–Sher thesis does not require that there be any connections among the ways a logical operation acts on domains of different sizes. Thus, it would permit a logical connective which acts like disjunction when the size of the domain is an even successor cardinal, like conjunction when the size of the domain is an odd successor cardinal, and like a biconditional at limits." (McGee 1996: 577)

(Feferman 1999: 38)

For Feferman, this point is more compelling than the other two:

For me, point 3 is perhaps the strongest reason for rejecting the Tarski–Sher thesis, at least as it stands

(Ibid.)

But his objection concerns only the sufficiency part of the Tarski–Sher thesis:

I agree completely [that] the Tarski–Sher thesis [is] a necessary condition for an operation to count as logical.

(Ibid., inversed sentence structure)

Still, it is a clear and strong criticism:

I . . . believe that if there is to be an explication of the notion of a logical operation in semantic terms, it has to be one which shows how the way an operation behaves when applied over one domain $M_0$ connects naturally with how it behaves over any other domain $M'_0$.

(Ibid.: 38–9)

As "a first step in that direction" Feferman proposes a revision of the Invariance-under-Isomorphism criterion. The revision consists in replacing "Isomorphism" by "Homomorphism," the resulting concept of logical operation being that of a "homomorphism invariant operation" (ibid.: 39). I will examine Feferman's proposal below, but first let me consider his criticisms.

## CONSIDERATION OF FEFERMAN'S CRITICISMS

I will begin by putting Feferman's criticisms in a proper perspective. There are a few significant points of similarity between Feferman's approach to the issues in question and mine (I prefer not to speculate about Tarski). First, Feferman does not question either the need for a criterion of logicality or the appropriateness of the semantic method for such a criterion. Second, Feferman regards the issue of logicality as a foundational issue, and is not averse to the pursuit of foundational studies. (On the contrary; Feferman has been extensively engaged in important foundational work, two examples of which are Feferman 1993*a* and Feferman and Hellman 2000.) Furthermore, Feferman's approach is neither logicist nor Platonist, conventionalist, intuitionist, or indispensabilist, but he is seeking a new approach to the foundations of logic and mathematics. (See, e.g., Feferman 1984, 1993*a*, and 1993*b*.) Finally, Feferman, as noted above, accepts the Invariance-under-Isomorphism criterion, as it stands, as a *necessary* condition on logicality, and his own proposal for a *sufficient* condition involves only a limited revision of that criterion. In light of these observations, I think it is reasonable to view Feferman's criticism as a restricted, internal criticism, rather than a full-scale external criticism. Nevertheless, this is a veritable criticism that requires careful consideration.

### Assimilation of logic to mathematics

A disagreement between a mathematician and a philosopher on the relation between logic and mathematics, such as that between Feferman and myself, was anticipated by Tarski:

> [T]he two possible answers [to the question whether mathematics is separate from logic] correspond to two different types of mind. A monistic conception of logic, set theory, and mathematics . . . appeals, I think, to a fundamental tendency of modern philosophers. Mathematicians, on the other hand, would be disappointed to hear that mathematics, which they consider the highest discipline in the world, is a part of something so trivial as logic; and they therefore prefer a development of set theory in which set-theoretical notions are not logical notions.
>
> (Tarski 1966: 153)[20]

[20] The ellipses and square-bracket formulations are partly intended to neutralize Tarski's tendency to identify the impact of the Invariance-under-Isomorphism/Permutation criterion on logic and mathematics with logicism. As pointed out earlier in this chapter, the new criterion leads to the "mathematization" of logic rather than to the "logicization" of mathematics. Although



But I think there is more to Feferman's position than a certain type of mind or a reverential attitude toward mathematics. In my view (as outlined above), mathematicians have a solid reason for regarding mathematics as dealing with non-logical notions, namely: their task. Their task (or one of their main tasks) is to discover and systematize the laws governing formal structures rather than apply these laws in discourse and reasoning. And the natural way for humans to study the laws governing a certain kind of structure is to construe these structures as structures of basic elements (of some kinds), i.e., in the case of formal structures, as structures of elements that do not satisfy the Invariance-under-Isomorphism criterion. But the two construals of formal objects do not conflict. To see this more clearly, let us draw an analogy to the conception of numbers in mathematical structuralism.

From the structuralist point of view there is no difference between studying the laws of arithmetic by studying a certain system of numbers or the corresponding system of sets. But to study the arithmetical laws the mathematician is best served by choosing some specific entities to work with, be they numbers or sets. From the point of view of the working number theorist, then, arithmetic is a theory of a specific kind of objects, but that does not conflict with the philosophical claim, reached by abstraction and generalization, that numbers are mere *places in a structure*, whose occupants' identity is immaterial.

In the case of formal operators, the notions mathematicians work with are, for the most part, lower-order, non-logical notions, while the notions logicians work with are, for the most part, logical notions, obtained from lower-order, non-logical, mathematical notions by "raising" them to a higher-order. It is this raising that captures their nature as formal or structural elements, and the laws governing them as laws of formal structure. Together, these two perspectives systematize our idea of formality.[21]

So we see that Feferman's justified claim that there are significant differences between logic and mathematics *is in fact satisfied* by the Invariance-under-Isomorphism criterion, especially on the formalist interpretation I have given to it here and in Sher (1991).

Feferman's criticism, however, raises other issues as well, some directly, others indirectly. One issue it raises indirectly is the role of common-sense intuition, or "gut feelings," in determining the relation between logic and mathematics. On this issue, I am afraid, we are in disagreement, since in my view the relation between logic and mathematics has very little to do with gut feelings. It is true that in approaching this issue, and in various stages of pursuing it, we use everyday intuition. But once we approach it as a theoretical issue, as we do when we construct a rigorous criterion of

either way mathematics and logic are one, the direction of reduction is philosophically significant: logicism attributes to mathematics the properties usually associated with logic, while mathematicism attributes to logic the properties usually associated with mathematics.

[21] An analogy with equivalence classes in mathematics might be helpful here. In some cases a given idea is better expressed by an equivalence class than by any of its constituent classes. But an equivalence class could not express this idea without its constituent classes, which are generally not equivalence classes, exemplifying it. In that sense, there is a division of labor between equivalence- and non-equivalence-classes in expressing that idea. This, indeed, is a natural way to understand mathematical structuralism as well.

logicality and develop a systematic account of logic to go with it, the role of gut-feelings becomes very limited. In fact, Feferman himself regards foundational studies as having a largely *theoretical* role: namely, "*conceptual clarification; interpretation [and] reduction . . . of problematic concepts and principles; organizational . . . foundations; and reflective expansion of concepts and principles*" (Feferman 1993*b*: 106). As such they are entitled to results that conflicts with some of our "gut feelings."

Since the Invariance-under-Isomorphism criterion, combined with the formalist account of logic, offers an informative and systematic account of the concept of logical operator, solves serious conceptual problems (e.g., with the definition of logical consequence), explains the relation between logic and truth, elucidates the role of logic in our system of knowledge, critically establishes many of the intuitive attributes of logic, and offers a substantive and methodologically economical account of the relation between logic and mathematics, it should not be judged based on "gut feeling."

Another issue raised by Feferman's criticism is ontological commitment. Feferman upholds the traditional view that logic, unlike mathematics, should have no ontological commitments. By assimilating logic to mathematics, he claims, the Invariance-under-Isomorphism criterion burdens it with considerable ontic commitments. By this Feferman means one of two things: (i) the fact that we resort to a set theoretical background language carries with it ontological commitments to sets; (ii) the enormous expressive power of the logic sanctioned by that criterion carries commitments to many ontologically-laden set theoretic theses. Clearly (i) is common to standard first-order logic and the logic sanctioned by our criterion. So let us turn to (ii). Consider the sentence:

$$(\exists!2^{\aleph_0}x)x = x \equiv (\exists!\aleph_1 x)x = x,$$

This is a well-formed sentence of the logic sanctioned by the Invariance-under-Isomorphism criterion, but for its truth-value in uncountable models to be determined, logic must be committed either to the continuum hypothesis (CH) or to its negation (∼CH). Does this saddle logic with the same ontological commitments as those of mathematics?

To get a first inkling of the difference between logical and mathematical commitments, consider the difference between the way the logical CH and the mathematical CH behave under negation. (This is a theme known from comparisons of first- and second-order CH; see, e.g., Shapiro 1991.) Let us call the mathematical statement expressing CH "$CH_M$" and the logical statement expressing CH "$CH_L$". Then, whereas ∼CH is captured by "$\sim CH_M$", it is not captured by "$\sim CH_L$". "$\sim CH_M$" can be added to set theory as an axiom without rendering set-theory inconsistent. But "$\sim CH_L$" cannot be a logical law, since logic—both standard logic and the logic sanctioned by the Invariance-under-Isomorphism criterion—has countable models (in which CH is trivially satisfied), and these would prevent it from being true in all models.

The main point is that while mathematics has direct ontological commitments, logic's ontological commitments are for the most part indirect. Aside from a few direct technical commitments—for example, a commitment to the existence of at least one individual (given the technical requirement that a model have a non-empty universe)—logic has only indirect ontological commitments, namely, commitments through its background theory of formal structure. And even these commitments are not existential in the usual sense; rather, they are commitments to the *formal possibility* of existence. Thus, as an axiom within (mathematical) set theory, Infinity says that an infinite set actually exists, but as an axiom within a background theory for logic, it says that an infinite structure of objects is *formally possible*.[22]



## Non-robust logical notions

Feferman notes that many of the logical operators sanctioned by the Invariance-under-Isomorphism criterion are not "robust" and argues that only "robust" operators should be classified as logical. The word "robust" can be interpreted in many ways, but Feferman has a specific interpretation in mind: for an operator defined in set-theoretical terms to be robust is to have "the same meaning independently of the exact extent of the set-theoretical universe" (cited above). And this idea, Feferman suggests, is captured by the set-theoretical concept of "absoluteness": to be robust is to be "absolute" (in the set-theoretic sense). The set-theoretic concept of ***absoluteness*** was introduced by Gödel in the course of proving the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis. His proofs involved the claim that in the "constructible universe" V = L (i.e., L exhausts the whole universe of sets). And to establish this claim Gödel used absoluteness results, whose basic concept can be defined as follows:

***A formula*** $\Phi(x_1, \ldots x_n)$ ***is absolute*** from a transitive class M to a transitive subclass N *iff* $\forall x_1 \ldots x_n[x_1, \ldots, x_n \in N \supset (N \models \Phi \equiv M \models \Phi)]$.

Gödel was especially interested in formulas which are absolute from V to L, and in particular, in the fact that the operation of forming all the "constructible" (definable) subsets of a given set is absolute from V to L. (See Gödel 1940 and discussion in Solovay 1990.) But the concept of absoluteness has been generalized in various ways, leading to many new applications.[23]

From the point of view of Feferman's criticism of the Invariance-under-Isomorphism criterion, the most relevant feature of the absoluteness requirement is that it does not allow operators to change their meaning by expansion or contraction of a given universe. This requirement renders "finite" an absolute operator (relative to ZFC) but "uncountable" not. A subset of the universe that satisfies "is finite" in a smaller

[22] I briefly discussed this matter in Sher (1996*a*: 682) where I pointed out additional references. Note: I do not mean to say that including CH, for example, as an axiom of our background theory of logic does not actually commit that theory to provide the bijections needed to secure the fact that the size of the continuum is aleph-1. What I mean to say is that if CH is included as an axiom of this theory, then ***it represents a formal law whose scope is the totality of formally possible structures of objects***. If we include CH in this theory, we are of course actually committed (not "possibly committed") to the existence of the requisite bijections (or of something else that will do the same job). I would like to thank Denis Bonnay for raising this issue.

[23] See, e.g., Burgess (1977), Väänänen (1985), and Tourlakis (2003).

model of set theory also satisfies it in a larger model (and vice versa, assuming it is included in the smaller model), but a subset of the universe that satisfies "is uncountable" in a small model (for example, a Löwenheim-Skolem model) does not satisfy it in a standard model. Accordingly, the quantifier "finitely many" is absolute, but "uncountably many" is not. But both quantifiers are logical according to the Invariance-under-Isomorphism criterion. Therefore, this criterion must be rejected, or so Feferman says.

In responding to Feferman's second criticism, I will first show that this criticism is weaker than it may seem to be, and then I will question the relevance of absoluteness to logicality.

(A) First, it should be pointed out that Feferman's criticism is directed at an *artifact* of a particular background theory we use to formulate the Invariance-under-Isomorphism criterion, but the idea underlying this criterion is not wedded to this, or any other background theory. In particular, the conception of logicality as formality, and even the conception of formality as invariance under 1–1 replacements of individuals, is not inherently connected to a particular set-theoretical language for which the question of "absoluteness" arises.

But even assuming this background language, Feferman's criticism is weaker than it may seem to be. Whereas in one sense the operator "uncountably many" changes its meaning from universe to universe, in another, more relevant sense, it does not. Let me explain. Clearly, as defined in a first-order set-theory—call it "T"—the predicate "x is uncountable" is satisfied by some countable set (i.e., an individual *b* to which countably many individuals *a* stand in the relation "x is a member of y") in some model of T. But the quantifier "there are uncountably many" is not satisfied by any countable set (a collection of countably many individuals) in any model of a first-order logical system in which it serves as a logical quantifier. To see this, the reader has to know how such a logical system is constructed, and this is something I have not discussed here. (A relevant discussion appears in chapter 3 of Sher 1991.) But let me try to explain the general principle underlying this claim briefly.

Consider the following:

In a first-order set-theory, $T_1$, we cannot see that the predicate "x is uncountable" (of $T_1$) is satisfied by a countable set in some model of $T_1$. To see that it is, we have to go to another theory, $T_2$, which is at least as strong (in the relevant sense) as $T_1$ and in which we can truly say that the formula "x is uncountable" of $T_1$ is satisfied by some countable[24] set in some model of $T_1$. Intuitively, from the point of view of $T_2$ the $T_1$-predicate "x is uncountable" is not robust, but from the point of view of $T_1$ it is.

Now, it is an essential feature of a logic L that the following are all done on the same level of discourse, or within the same background theory—call it "T1": (i) the definitions of the logical constants of L, (ii) the definitions of the operators corresponding to them, (iii) the definition of the models of L, (iv) the definition of "true in a model," (v) the definition of "countable" and "uncountable," (vi) the definition of robustness, etc. From the above considerations it follows that from the point of view of T1 the logical quantifier "there are uncountably many" of L has a fixed meaning and as such is robust. This is expressed by the fact that (from the point of view of T1) the Löwenheim–Skolem theorem does not hold the logical constants of L: "(For uncountably many x) x = x" has no countable models. (Of course, from the point of view of yet another theory, T2, T1 itself may be subject to the Löwenheim–Skolem theorem. But from the point of view of T2, "robustness" may have a non-standard meaning as well, as Feferman noted.)

[24] i.e., countable from the point of view of $T_2$.



(B) Absoluteness is an interesting and in certain respects a desirable property, but should we restrict our concept of logicality to operators satisfying this property? To put things in perspective, there are many interesting and desirable properties we don't restrict our concepts to. Take, for example, *decidability.* Decidability is an interesting and desirable property of logics, yet we do not restrict ourselves to decidable logics. The price of setting decidability as an upper boundary on our concept of logic is simply too high. Clearly, sentential logic or even monadic standard first-order logic is too narrow to exhaust our concept of logic or even to serve as a working logic for mathematics. Or consider *completeness.* Completeness is a desirable property of theories. But we would have to remove most of mathematics from the realm of axiomatized first-order theories if we were to require that only complete axiomatizations be permitted in that realm. Saying that *generally* only complete theories are genuine theories would be even more absurd.

The question arises whether the same does not hold for absoluteness. It clearly does in some cases. For example, we cannot restrict set theory to absolute concepts, since this would involve omitting many of its most basic concepts, e.g., the concept of cardinality. But does it hold in the case of logic?

Let us first see how the formalist conception of logic answers this question. From the point of view of this conception, logic requires a background theory of formal structure, and it is an open question what the best theory of formal structure is. In principle we are looking for the most economical theory that is sufficiently strong to account for formal structures in a comprehensive manner. Two more economical candidates than ZFC are ZFC + (V = L) and Feferman's predicative system, but there are other candidates as well, and the jury is still out on what the best available theory is. However, absoluteness per se is not a reasonable constraint on a theory of formal structure, since *a property is absolute iff it is insensitive to a certain formal difference between universes* (namely, the difference between *larger* universes and *smaller* universes *included* in them). This means that a theory that admits only absolute notions neglects some formal differences between objects, and as such is not an acceptable theory of *formal structure.*

These considerations show that: (i) the fact that absoluteness is desirable for some purposes does not mean that it is appropriate for the purpose of constructing a criterion of logicality; and (ii) to accept absoluteness as a constraint on logicality we need to renounce the formalist conception of logic and the associated justification of our criterion of logicality. I would be very interested to examine a philosophical conception of logic that fits in with the absoluteness requirement and offers a foundational justification of a concept of logicality satisfying it. As far as I know, none is available yet.

## Operators with "non-uniform meaning", "split identity", or "unnatural behavior"

Feferman's main objection to the Invariance-under-Isomorphism criterion is that it sanctions logical operators lacking a unified identity, or a natural connection between the way they behave in different universes, or (when we consider the terms denoting them) the same meaning in different universes. Such operators can behave one way over universes of cardinality $\kappa$ and another way over universes of cardinality $\lambda(\neq \kappa)$, i.e., their meaning, or identity, depends on the size of the universe, and there is no natural connection between the way they behave in universes of size $\kappa$ and universes of size $\lambda$.

Before considering Feferman's criticism, it would be instructive to note that his particular example of such an operator is in fact *not* countenanced by my version of the "Tarski–Sher" thesis. Feferman's example is that of a *propositional connective,* O, "which acts like disjunction when the size of the domain is an even successor cardinal, like conjunction when the size of the domain is an odd successor cardinal, and like a biconditional at limits" (cited above). I agree with Feferman's claim that O is not a proper logical operator, but not with his reason for claiming so. Propositional connectives should not depend on the size of the universe (of individuals) because this has nothing to do with *truth-functionality.* The problem with O, as I see it, is that as a *propositional* operator it should not take into account *universes of individuals* at all. And in my version of the logicality criterion propositional operators do not. Propositional operators (connectives) are defined in terms of propositional rather than objectual structures, and propositional structures have a universe of propositions rather than a universe of individuals. Indeed, they take into account only one universe —the universe of all propositions. The operator mentioned in Feferman's example is therefore not logical according to my (version of the) "Tarski–Sher" logicality criterion.[25]

But the phenomenon Feferman talks about is true of other operators satisfying this criterion. Take the objectual operator Q defined by: Given a universe A and a subset B of A,

$Q_A(B) = T$ *iff either* A *is countable and* B = A *or* A *is uncountable and* B is *not empty*.

[25] 1. See p. 322–4 above.
2. I should indicate that I had the opportunity to correct Feferman's error when I received a pre-publication copy of McGee (1996) from which this example is taken, but I failed to do so, since in the context of McGee's chapter it seemed an insignificant point. In the present context, however, it is more significant.



Then Q behaves like ∀ in countable universes and like ∃ in uncountable universes.[26] Is the fact that Q has this "split" identity a good reason for refusing to count it as a logical operator? In answering this question I will make a few points:

(A) To the extent that we refuse to count Q as a logical operator because it is an "unnatural" operator, it should be noted that numerous unnatural objects (properties, relations, functions) are widely accepted in other fields. Feferman himself (2000) brings numerous examples of what he calls "monstrous" or "pathological" objects that are generally accepted by mathematicians (he included).

Indeed, even in standard logic there are many "unnatural" operators, including logical operators of "split" identity or meaning, operators which do not seem "to have the same meaning" or "be the same operators" in different settings. Two examples would suffice:

(a) A 132-place propositional connective, C, such that:

(i) C behaves like a 132-place Conjunction in rows with 0–23 T's,
(ii) C behaves like a 132-place Disjunction in rows with 24–79 T's, and
(iii) C behaves like the Majority Connective in all other rows.

(b) A quantifier Q*, *definable in standard first-order logic,* such that:

(i) Q* behaves like "All" (∀) in universes of cardinality <101,
(ii) Q* behaves like "Some" (∃) in universes of cardinality 101–745, and
(iii) Q* behaves like "None" (∼∃) in universes of all other cardinalities.

Intuitively, the identity (or meaning) of these operators is no less "split" than that of Q, yet they are accepted as legitimate logical operators by most logicians and philosophers (including Feferman, I am sure). Why, then, should we discriminate against Q? In what way is Q *less natural,* or its identity or meaning *more "split",* than those of C and Q*, which we all accept as legitimate logical operators?

(B) My point is not that there is no value or interest in a specific concept of "natural operator" (or "natural connection between an operator's behavior in different universes"), but such a concept has nothing much to do with our idea of *logicality.*

We may wish to distinguish "natural" logical operators from "unnatural" logical operators or "natural" operators in general from "unnatural" operators in general,

[26] This is not the only kind of operator that exhibits this phenomenon. In connection with my claim that propositional connectives are *not* sensitive to the size of objectual universes, I would like to clarify that some logical operators *defined in terms* of objectual functions corresponding to the logical connectives (union, intersection, etc.) are sensitive to the size of universes of individuals, but *these operators themselves do not correspond to any propositional connective.* Thus, the functional operator F, defined, for an objectual universe A and subsets B and C of A, by:

(i) $F_A(B, C) = B \cap C$ if A is countable;
(ii) $F_A(B, C) = B \cup C$ if A is uncountable

is a logical operator according to the Invariance-under-Isomorphism criterion, but it is not the objectual equivalent of *any* propositional connective.

in the same way that we may wish to distinguish "natural" functions from "unnatural" functions or "natural" relations in general from "unnatural" relations in general. But just as the latter would not undermine, or force us to change, our criterion of a functionality (of a relation being functional), so the former would not undermine, or force us to change, our criterion of a logicality (of an operator being logical).

We could impose on ourselves a "naturalness" constraint in choosing a logical system to work with, but this would be a separate constraint from the "logicality" constraint we would impose on such a system.

(C) Finally, there is a strong unity (or uniformity) to the concept of *logical operator* delineated by the Invariance-under-Isomorphism criterion and a clear concept of *same logical operator* associated with it. Both are generated by our interpretation of this criterion as a criterion of *formality:* All and only formal operators are logical, and each logical operator describes one way in which an operator takes into account some formal features of a given situation. Thus all logical operators are *unified* in being formal, and a logical operator is *the same* in different universes iff there is some formal pattern of objects-having-properties-and-standing-in-relations-within-situations that its trajectory through the different universes represents. Since the size of the universe is a basic formal feature of objectual situations, it is—and should be—a central parameter of some objectual formal operators.

## FEFERMAN'S CRITERION OF LOGICALITY

Feferman's criterion of logicality is proposed "as a first step" in the "direction" of showing "how the way an operation behaves when applied over one domain $M_0$ connects naturally with how it behaves over any other domain $M_0'$" (Feferman 1999: 38–9). It is obtained from the Invariance-under-Isomorphism criterion by replacing "isomorphism" by "homomorphism" (or what is sometimes called "strong homomorphism"), i.e., by replacing the requirement that logical operators be invariant under any *1–1* and *onto* transformations of structures by the requirement that they be invariant under any *onto* transformations of structures. We can formulate Feferman's criterion as follows:

**Invariance-under-Homomorphism:** *An operator O is logical iff it is invariant under all homomorphisms of its argument-structures,*

where:

(i) A structure, $\langle A, \beta_1, \ldots, \beta_n \rangle$, is *homomorphic* to a structure $\langle A', \beta_1', \ldots, \beta_k' \rangle$ iff $n = k$ and there is a surjection $f$ from A to A' such that for every $1 \leq i \leq n$, $\beta_i'$ is the image of $\beta_i$ under $f$,

(ii) An n-place operator O is invariant under all homomorphisms of its argument-structures iff for any of its argument-structures, $\langle A, \beta_1, \ldots, \beta_n \rangle$ and $\langle A', \beta_1', \ldots, \beta_n' \rangle$: if $\langle A, \beta_1, \ldots, \beta_n \rangle$ is homomorphic to $\langle A', \beta_1', \ldots, \beta_n' \rangle$, then $O_A(\beta_1, \ldots, \beta_n) = O_{A'}(\beta_1', \ldots, \beta_n')$.



The effects on the *generality* and *formality* of our concept of logicality are: (a) Since every bijection is a surjection but not vice versa, there are more surjections than bijections, and Invariance-under-Homomorphisms (surjections) is invariance under more transformations of structures. As a result, the concept of logicality associated with the new criterion is more *general* than that associated with the old criterion. All logical operators under the former are logical under the latter, but not vice versa. The new criterion, however, does not render logic maximally *general* (it does not require logical operators to be invariant under all transformations whatsoever); therefore the concept of logicality associated with it cannot be fully explained or justified in terms of *generality.* (b) Since surjections overlook certain gaps in size between their domain-universe and their range-universe (mapping larger universes into smaller ones), invariance under surjections does not respect an important *formal* difference between structures, namely, difference in size or cardinality. As a result, the new criterion leads to a concept of logicality that parts ways with that of *formality,* making the explanation and justification of the old criterion inaccessible to it.

The new criterion, however, may be thought to satisfy Feferman's requirement that logical operators "behave in the same way in all universes." Intuitively, a homomorphism is a mapping *h* such that the distinguished elements of the smaller structure are obtained from those of the larger one by ' "shrinking along' *h*" (ibid.: 39), and this "shrinking" explains the sense in which an operator preserves its identity when moving from larger universes to a smaller ones.

Is *Invariance-under-Homomorphism* a reasonable criterion of logicality? To help us answer this question let us point at a few significant examples of operators that do and do not satisfy it.

(a) *Isomorphism-invariant operators that are also homomorphism-invariant:*

(i) The operators corresponding to the logical connectives.

(ii) The existential and universal quantifiers.

(iii) The quantifier "is well-founded" (whose arguments, in any given universe, are the binary relations on that universe).

(b) *Isomorphism-invariant operators that are* not *homomorphism-invariant:*

(i) The (standard) Identity relation.

(ii) Cardinality quantifiers (including finite-cardinality quantifiers like "There are exactly 5" and infinite-cardinality quantifiers like "There are uncountably many").

(iii) The monadic quantifier "Most" (as in "Most things are B").

(iv) Quantifiers that behave like one familiar quantifier in universes of certain cardinalities and like a different familiar quantifier in universes of other cardinalities (for example, Q of the last section).

These examples suggest that the Invariance-under-Homomorphism criterion gives rise to a "hybrid" logic. This logic coincides neither with standard first-order logic nor with our "formal" logic, yet it is not intermediate between the two either, since in certain ways it is weaker than standard first-order logic. In particular, neither the identity relation (=) nor the finite-cardinality quantifiers ("There are at-least/exactly/at-most n things such that") of standard first-order logic satisfy it. At the same time it is stronger than standard first-order logic since it is satisfied by such non-standard quantifiers as the well-foundedness quantifier.

Feferman does not fully embrace the Invariance-under-Homomorphism criterion as a criterion of logicality. Rather, having "been moving more and more to the position that the classical first-order predicate logic has a privileged role in our thought" (ibid.: 32), he is looking for ways to adjust it so it classifies all and only the standard logical operators as logical. His investigations first lead to an adjustment that, assuming Invariance-under-Homomorphism is so formulated as to apply to objectual operators only, could be expressed by:

**Adjusted Invariance-under-Homomorphism criterion (I):**

*A first-order operator is logical iff it is:*

*either (i) a monadic quantifier satisfying the Invariance-under-Homomorphism criterion,*

*or (ii) a truth-functional connective,*

*or (iii) an operator definable from logical operators within the λ-calculus.*

By formulating the Invariance-under-Homomorphisms criterion in such a way that it applies to propositional connectives as well, however, Feferman obtains a more unified version of this adjusted criterion:

**Adjusted Invariance-under-Homomorphism criterion (II):**

*A first-order operator is logical iff it is:*

*either (i) a monadic quantifier satisfying the Invariance-under-Homomorphism criterion,*

*or (ii) a propositional operator (monadic or not monadic) satisfying this criterion,*

*or (iii) an operator definable from logical operators within the λ-calculus.*

The adjusted criterion differs from the original Invariance-under-Homomorphism criterion in setting a *type restriction* on logical quantifiers: only *monadic* first-order quantifiers—quantifiers of the type O(B), where B is a subset of a given universe—and not first-order quantifiers of any other type—i.e., relational or polyadic quantifiers—are logical. That is, only monadic quantifiers are subject to the Invariance-under-Homomorphism test. (Linguistically, this restricts us to quantifiers of the form "(Qx)Px", ruling out in advance, i.e., prior to applying the Invariance-under-Homomorphism criterion, all relational quantifiers (e.g., "$Most^2$," as in "Most B's are C's") and polyadic quantifiers (e.g., "Is a well-ordering").)

This restriction yields *almost* the desired result: all and only the logical operators of standard first-order logic *without identity* are logical.

What about Identity? In considering this question Feferman says:

> It is undeniable that the relation of identity has a "universal", accepted, and stable logic (at least in the presence of totally defined predicates and functions, as is usual in PC with =), and that argues for giving it a distinguished rule in logic even if it should not turn out to be logical on its own under some cross-domain invariance criterion, such as under homomorphisms.
>
> (Ibid.: 44)



To include identity as a logical operator we can simply *postulate* that it is, closing logical operators under definability as before. We thus get the third version of the adjusted criterion:

**Adjusted Invariance-under-Homomorphism criterion (III):**

*A first-order operator is logical iff it is:*

*either (i) a monadic quantifier satisfying the Invariance-under-Homomorphism criterion,*

*or (ii) a propositional connective satisfying this criterion,*

*or (iii) the identity relation,*

*or (iv) an operator definable from logical operators within the λ-calculus.*

This criterion classifies identity and the finite-cardinality quantifiers as logical, thus providing a characterization of the standard first-order logical operators as logical.

Is either the Invariance-under-Homomorphism criterion or the Adjusted Invariance-under-Homomorphism criterion (in any of its versions) an adequate criterion of logicality?

Van Benthem (2002) and Bonnay (2008) point out that the Invariance-under-Homomorphism criterion is subject to Feferman's first two criticisms—assimilation of logic to mathematics and non-robust logical operators—and as such is inadequate from his own perspective. I would add that by affirming the logicality of the finite-cardinality quantifiers—including "split identity/meaning" finite-cardinality quantifiers (those whose behavior in universes of different sizes is "unnaturally connected")—Feferman's third adjusted criterion also violates the third criticism. Finally, Bonnay (2008) criticizes the *ad hoc* nature of Feferman's restriction of logical quantifiers to monadic ones in the adjusted versions of his criterion.[27]

Most of these criticisms, however, do not speak against Feferman's criteria from my point of view, since the "weaknesses" they talk about are no weaknesses at all from my perspective. The one exception is the *ad hocness criticism*, which points to what, in my view, is the main challenge to any criterion of logicality, namely, a solid philosophical justification, which is missing from Feferman's discussion, and indeed not even attempted by him. That such a justification needs pursuing is also Feferman's view of the matter:

> Whether that [i.e., the notion of a logical operation as "definable from homomorphism-invariant monadic operations"] (or any other invariance notion) can be justified on fundamental conceptual grounds is . . . in need of pursuit.
>
> (Feferman 1999: 32)

[27] Feferman's tries to justify this restriction linguistically, by appealing to a linguistic conjecture which says that most non-monadic quantifiers used in natural language are "lifted" in one way or another from monadic quantifiers (Keenan and Westerståhl, 1997). But this conjecture is restricted to natural-language applications, is not strictly universal, is (at least as of now) unsubstantiated, and assumes the logicality of monadic quantifiers that Feferman rejects. More importantly, it is not clear that linguistic support of a logical-philosophical restriction is of much relevance.

This is a good note on which to end. I must add, however, that there exist other serious proposals for revision of the Invariance-under-Isomorphism criterion. These include Peacocke (1976), McCarthy (1981), MacFarlane (1991), Bonnay (2008), and Casanova's (2007), and they each require a careful consideration.

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