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The Bounds of LogicGila SherA Generalized Viewpoint

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Contents

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Chapter 1 New Bounds? 1

Chapter 2 The Initial Generalization 10

Chapter 3

To Be a Logical Term 36

Chapter 4 Semantics from the Ground Up 67

Chapter 5 Ways of Branching Quantifiers 105

Chapter 6

A New Conception of Logic130Appendix141Notes149References159Index of Notation169Index of Terms175

For Peter

Whatever the fate of the particulars, one thing is certain. There is no going back to the view that logic is [standard] first-order logic. Jon Barwise, *Model-Theoretic Logics*

When I went to Columbia University to study with Prof. Charles Parsons, I felt I was given a unique opportunity to work on "foundational" issues in logic. I was interested not so much in the controversies involving logicism, intuitionism, and formalism as in the ideas behind "core" logic: first-order Fregean, Russellian, Tarskian logic. I wanted to understand the philosophical force of logic, and I wanted to approach logic critically.

Philosophical investigations of logic are difficult in that a fruitful point of view is hard to find. My own explorations started off when Prof. Parsons pointed out to me that some mathematicians and linguists had generalized the standard quantifiers. Generalization of quantifiers was something I was looking for since coming upon Quine's principle of ontological commitment. If we understood the universal and existential quantifiers as particular instances of a more general form, perhaps we would be able to judge whether quantification carries ontological commitment. So the idea of generalized quantifiers had an immediate appeal, and I sat down to study the literature.

The generalization of quantifiers gives rise to the question, What is logic? in a new, sharp form. In fact, it raises two questions, mutually stimulating, mutually dependent. More narrowly, these questions concern quantifiers, but a broader outlook shifts the emphasis: What is it for a *term* to be logical? What are *all* the terms of logic? Sometimes in the course of applying a principle, we acquire our deepest understanding of it, and in the attempt to extend a theory, we discover what drives it. In this vein I

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thought that to determine the full scope of logical terms, we have to understand the idea of logicality. But the actual expansion of quantifiers gives us hands-on experience that is, in turn, valuable in tackling "logicality." Prof. Parsons encouraged me to select this as the topic of my dissertation, and *The Bounds of Logic*, a revised version of my thesis, follows the course of my inquiries.

The idea of logic with "generalized" quantifiers has, in the last decades, commanded the attention of mathematicians, philosophers, linguists, and cognitive scientists. My own perspective is less abstract than that of most mathematicians and less empirical than the viewpoint of linguists and cognitive scientists. I decided not to address the logical structure of natural language directly. Instead, I would follow my philosophical line of reasoning unmitigatedly and then see how the theory fared in the face of empirical data. If the reasoning was solid, the theory would have a fair chance of converging with a sound linguistic theory, but as a philosophical outlook, it should stand on its own.

The book grew out of three papers I wrote between 1984 and 1987: "First-Order Quantifiers and Natural Language" (1984), "Branching Quantifiers, First-Order Logic, and Natural Language" (1985), and "Logical Terms: A Semantic Point of View" (1987). These provide the backbone of chapters 2, 5, and 3. Chapter 1 is based on my thesis proposal, and the ideas for chapter 4 were formulated soon after the proposal defense. "Logical Terms" was rewritten as "A Conception of Tarskian Logic" and supplied with a new concluding section. This section, with slight variations, constitutes chapter 6. An abridged version of "A Conception of Tarskian Logic" appeared in *Pacific Philosophical Quarterly* (1989a). I would like to thank the publishers for their permission to reproduce extensive sections of the paper. "Branching Quantifiers" gave way to "Ways of Branching Quantifiers" and was published in *Linguistics and Philosophy* (1990b). I am thankful to the publishers of this journal for allowing me to include the paper (with minor revisions) here.

At the same time that I was working on my thesis, other philosophers and semanticists were tackling tangential problems. In general, my guideline was to follow only those leads that were directly relevant. A few related essays appeared too late to affect my inquiry. In the final revision I added some new references, but for the most part I did not change the text. I felt that the original conception of the book had the advantage of naturally leading the reader from the questions and gropings of the early chapters to the answers in the middle and from there to the formal developments and the philosophical ending. Preface

There is, however, one essay that I would like to mention here because it is so close to mine in its spirit and its view on the scope of logic. This is Dag Westerståhl's unpublished dissertation, "Some Philosophical Aspects of Abstract Model Theory" (1976), which I learned of a short time before the final revision of this book was completed. Had I come upon it in the early stages of my study, I am sure it would have been a source of inspiration and an influence upon my work. As it turned out, Westerståhl relied on a series of "intuitions about logic," while I set out to investigate the bounds of logic as a function of its goal, drawing upon Tarski's early writings on the foundations of semantics.

Chapter by chapter, I proceed as follows: In chapter 1, I set down the issues the book attempts to resolve and I give an outline of my philosophical approach to logic. Chapter 2 analyzes Mostowski's original generalization of quantifiers, tracing its roots to Frege's conception of statements of number. The question then arises of how to extend Mostowski's work. I discuss a proposal by Barwise and Cooper (1981) to create a system of *nonlogical* quantifiers for use in linguistic representation. Pointing to weaknesses in Barwise and Cooper's approach, I advocate in its place a straightforward extension of the *logical* quantifiers, as in Lindström (1966a), and show how this can be naturally applied in natural-language semantics. It is not clear, however, what the philosophical principle behind Mostowski's work is. To determine the scope of logical quantifiers in complete generality, we need to analyze the notion of "logicality." This leads to chapter 3.

For a long time I thought I would not be able to answer the questions posed in this work. I would present the issues in a sharp and, I hoped, stimulating form, but as for the answers, I had no idea what the guiding principle should be. How would I know whether a given term, say "being a well-ordering relation," is a logical term or not? What criterion could be used as an objective arbiter? The turning point for me was John Etchemendy's provocative essay on Tarski. Etchemendy's charge that Tarski committed a simple fallacy sent me back to the old papers, and words that were too familiar to convey a new meaning suddenly came to life. My answer to the question of logicality has three sides: First, it is an analysis of the ideas that led Tarski to the construction of the syntactic-semantic system that has been a paradigm of logic ever since. Second, it is an argument for the view that the original ideas were not fully realized by the standard system; it takes a far broader logical network to bring the Tarskian project to true completion. Finally, the very principles that underlie modern semantics point the way to a simple, straightforward criterion of

logicality. I spell out this criterion and I discuss the conception of logic that ensues. As a side note I should say that although chapter 3 was not written as a defense of the historical Tarski, it contains, I believe, all that is needed to prove the consistency of Tarski's approach.

Chapter 4 presents a formal semantics for the "unrestricted" first-order logic whose boundaries were delineated in chapter 3. The semantic system is essentially coextensional with Lindström's, but the method of definition is constructive—a semantics "from the ground up." What I try to show, first informally and later formally, is how we can build the logical terms over a given universe by starting with individuals and constructing the relations and predicates that will form the extensions of logical terms over that universe. Chapter 4 also investigates the enrichment of logical vocabulary as a tool in linguistic semantics, pointing to numerous applications and showing how increasingly "stronger" quantifiers are required for certain complex constructions.

Chapter 5 was the most difficult chapter to write. Whereas the book in general investigates the scope and limits of logical "particles," this chapter inquires into new possibilities of combining particles together. My original intent was to study the new, "branching" structure of quantifiers to determine whether it belongs to the new conception of logic. But upon reading the literature I found that in the context of "generalized" logic it has never been determined what the branching structure really is. Jon Barwise's pioneer work pointed to several partial answers, but a general semantics for branching quantifiers had yet to be worked out. My search for the branching principle led to a new, broader account than was given in earlier writings. I introduce a simple first-order notion of branching, "independence"; I universalize the existent definitions due to Barwise; and I point to a "family" of branching structures that include, in addition to "independent," Henkin, and Barwise quantifiers, also a whole new array of logico-linguistic quantifier constructions.

In chapter 6, I draw several philosophical consequences of the view of logic developed earlier in the book. I discuss the role of mathematics in logic and the metaphysical underpinning of semantics, I investigate the impact of the new conception of logic on the logicist thesis and on Quine's ontological-commitment thesis, and I end with a proof-theoretical outlook. This chapter is both a summation and, I hope, an opening for further philosophical inquiries.

The bounds of logic, on my view, are the bounds of mathematical reasoning. Any higher-order mathematical predicate or relation can function as a logical term, provided it is introduced in the right way into the

Preface

xii

syntactic-semantic apparatus of first-order logic. Logic provides a special framework for formalizing theories, a framework that draws out their necessary and formal consequences. Every formal and necessary consequence is identified by some logic, and only necessary and formal consequences pass the test of logicality. This view is accepted in practice by many logicians working in "abstract" first-order logic. My view also stands in basic agreement with that of natural-language logicians. Extended logic has made a notable contribution to linguistic analysis. Yet logical form in linguistics is often constrained by conditions that have no bearing on philosophy. To my mind, this situation is natural and has no limiting effect on the scope of logic.

On the other side of the mat stand two approaches to logic: First and obviously, there is the traditional approach, according to which standard logic is the whole of logic. No more need be said about this view. But from another direction some philosophers see in the collapse of traditional logic a collapse of logic itself as a distinctive discipline. With this view I adamantly disagree. Logic is broader than traditionally thought, but that does not mean anything goes. The boundaries of logic are based on a sharp, natural distinction. This distinction serves an important methodological function: it enables us to recognize a special type of consequence. To relinquish this distinction is to give up an important tool for the construction and criticism of theories.

The writing of this work was a happy experience, and I am very thankful to teachers, colleagues, friends, and family who helped me along the way.

I was very fortunate to work with Charles Parsons throughout my years at Columbia University. His teaching, his criticism, the opportunity he always gave me to defend my views, his expectation that I tackle problems I was not sure I could solve—all were invaluable not only for this book but for the development of my philosophical thought. I am most grateful to him.

My first dissertation committee was especially supportive and enthusiastic, and I would like to thank Robert May and Wilfried Sieg for this and for their continuing interest in my work after they left Columbia. Robert May was actively involved with my book until its completion, and I am very thankful to him for his constructive remarks and for urging me to explore the linguistic aspect of logic. Isaac Levi and Shaugan Lavine joined my dissertation committee at later stages. Levi taught me at Columbia, and his ideas had an impact on my thought. I thank him for this and

xiv

Preface

for his conversation and support. Shaughan Lavine contributed numerous useful comments on my thesis, and I am very thankful to him.

John Etchemendy sent me his works on Tarski. These were most important in developing my own views, and I am grateful to him.

During the academic year 1987/1988 I was a visiting scholar at MIT, and I would like to thank the Department of Linguistics and Philosophy for its hospitality. I had interesting and stimulating conversations with George Boolos, Jim Higginbotham, Richard Larson, and Noam Chomsky, and I am particularly thankful to Richard Cartwright for his contribution to my understanding of Tarski.

While writing the dissertation I was teaching first at Queens College and later at Barnard College. I would like to thank the members of the two philosophy departments for the supportive environment. To Alex Orenstein, Sue Larson, Hide Ishiguro, Robert Tragesser, and Palle Yourgrau I am thankful for their conversation and friendliness.

The thesis developed into a book while I was at my present position at the University of California, San Diego. I am very grateful to my new friends and colleagues at UCSD for the stimulating and friendly atmosphere. I am especially indebted to Philip Kitcher for his conversation and advice. I am also very thankful to Oron Shagrir for preparation of the indexes.

Betty Stanton of Bradford Books, The MIT Press, encouraged me to orient the book to a wider audience than I envisaged earlier. I am very thankful for her suggestions.

As editor of *Linguistics and Philosophy* Johan van Benthem commented on my "Ways of Branching Quantifiers," and his comments, as well as those of two anonymous referees, led to improvements that were carried over to the book. I appreciate these comments. I am also thankful for comments by referees of The MIT Press.

Hackett Publishing Company allowed me to cite from Tarski's works. I am thankful for their permission.

I gave several talks on branching quantifiers, and I would like to thank the audiences at the Linguistic Institute (1986), MIT (1987), and the University of Texas at Austin (1990).

My interest in the philosophy of logic arose when I was studying philosophy at the Hebrew University of Jerusalem. I am grateful to my teachers there, especially Eddy Zemach and Dale Gottlieb, whose stimulating discussions induced my active involvement with issues that eventually led to my present work. I could not have written this book without the abounding support given to me by my family. My sons, Itai and Shlomi, were very young when I started by studies at Columbia University, and they grew up with my work. I thank you, Itai and Shlomi, for your patience with a busy mother, for your liveliness and philosophical thoughts, for rooting for me, and for much much more. My mother took over many of my duties during the last months of preparing the book. I am most grateful to her for this and for her support throughout my studies.

It is too late to thank my father, Shlomo Yoffe. He was a staunch rationalist who believed in straight, honest reasoning. He taught me to think and enjoy thinking. He respected intellectuals but not academic titles. His influence, more than anyone else's, led to my career in philosophy.

I am dedicating this book to my husband, Peter Sher, with love and gratitude. He followed my work closely from its beginning, and his suggestions regarding style and presentation of ideas left their marks on every page of this book. With great generosity he encouraged me to resume my studies in philosophy some time ago, and with great generosity he has supported me in my work since then.

xv

The Bounds of Logic

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New Bounds?

"Logic," Russell said, "consists of two parts. The first part investigates what propositions are and what forms they may have.... The second part consists of certain supremely general propositions which assert the truth of all propositions of certain forms.... The first part ... is the more difficult, and philosophically the more important; and it is the recent progress in this part, more than anything else, that has rendered a truly scientific discussion of many philosophical problems possible."¹

The question underlying this work is, Are generalized quantifiers a case in question? Do they give rise to new, philosophically significant logical forms of propositions "enlarging our abstract imagination, and providing \dots [new] possible hypotheses to be applied in the analysis of any complex fact"?² Does the advent of generalized quantifiers mark a genuine breakthrough in modern logic? Has logic, in Russell's turn of expression, given thought new wings once again?

Generalized quantifiers were first introduced as a "natural generalization of the logical quantifiers" by A. Mostowski in his 1957 paper "On a Generalization of Quantifiers"³. Mostowski conceived his generalized quantifiers semantically as functions from sets of objects in the universe of a model for first-order logic to the set of truth values, {truth, falsity}, and syntactically as first-order formula-building operators that, like the existential and universal quantifiers, bind well-formed formulas with individual variables to form other, more complex well-formed formulas. Mostowski's quantifiers acquired the name "cardinality quantifiers," and some typical examples of these are "there are finitely many x such that ...," "most things x are such that ...," etc.

Mostowski's paper opened up the discussion of generalized quantifiers in two contexts. The first and more general context is that of the scope and subject matter of logic. Although Mostowski declared that at least some

New Bounds?

2

generalized quantifiers belong in any systematic presentation of symbolic logic, this aspect, with the foundational issues it raises, was not thoroughly investigated either by him or by other mathematicians who took up the subject. The second, more specific context has to do with the properties of formal first-order systems with generalized quantifiers, particularly in comparison to "classical" first-order logic and its characteristic properties: completeness, compactness, the Löwenheim-Skolem property, etc. This was the main concern of Mostowski's research, and it became the focus of the ensuing surge of mathematical interest in the subject.⁴

In contrast to the extensive and prolific treatment that generalized quantifiers have received in mathematics, the philosophical yield has been rather sparse. The philosophical significance of generalized quantifiers was examined in a small number of contemporary papers by such authors as L. H. Tharp (1975), C. Peacocke (1976), I Hacking (1979), T. McCarthy (1981) and G. Boolos (1984b) as part of an attempt to provide a general characterization of logic and logical constants. The mathematical descriptions of generalized quantifiers and the numerous constructions by mathematicians of first-order systems with new quantifiers prompted the question of whether such quantifiers are genuinely logical. Although the discussions mentioned above are illuminating, they reach no definite or compelling conclusions, and, to the best of my knowledge and judgement, the question is still open.

To inquire whether "generalized" quantifiers are logical in complete generality, we have to ascend to a conceptual linguistic scheme that is independent of, or prior to, the determination of logical constants and, in particular, logical quantifiers. We will then be able to ask, What expressions in that linguistic scheme are logical quantifiers? What are all its logical quantifiers? The scheme has to be comprehensive enough, of course, to suit the general nature of the query.

A conceptual scheme like Frege's hierarchy of levels naturally suggests itself. In such a scheme the level of a linguistic expression can be determined prior to, and independently of, the determination of its status as a logical or nonlogical expression.⁵ And the principles underlying the hierarchy—namely the characterization of expressions as complete or incomplete and the classification of the latter according to the number and type of expressions that can complete them—are universally applicable. From the point of view of Frege's hierarchy of levels, the system of standard first-order logic consists of a first-level language plus certain second-level unary predicates (i.e., the universal and/or existential quantifiers) and

a "complete" set of truth-functional connectives. Our question can now be formulated with respect to Frege's linguistic scheme as follows: Which second-level predicates and relations can combine with first-level predicates, relations, functional expressions, proper names and sentential connectives to make up a first-order logic? What makes a second-level predicate or relation into a first-order logical quantifier? (analogously for higher-order quantifiers.)

To ask these questions is to investigate various criteria for logical "quantifierhood" with respect to their philosophical significance, formal results, and linguistic plausibility. At the two end points of the spectrum of possible criteria we find those that allow any second-level predicate and relation as a logical quantifier (this is possible within Frege's scheme because of the syntactic structure of second-level expressions) and those that allow only the universal and existential quantifiers as logical quantifiers. While the former amount to a trivialization of logic, the latter preclude extension altogether. For that reason, the area in-between is the most interesting for a critical investigator.

The formal scheme whose extension is considered in this book is Fregean-Russellian mathematical logic with Tarskian semantics. More specifically, the investigation concerns Tarski's model-theoretic semantics for firstorder logic. With respect to Tarski's semantics one may wonder whether it makes sense at all to consider its extension to a logic with new quantifiers. To begin with, we can see that the structure of a first-order Tarskian model does allow for a definition of truth (via satisfaction) for a language richer than that of standard first-order logic. One line of reasoning pointing to the naturalness of extension is the following: If Tarskian modeltheoretic semantics is philosophically correct, then a first-order model offers a faithful and precise mathematical representation of truth (satisfaction) conditions for a first-order extensional language. The formal correctness of this semantics ensures that no two distinct (i.e., logically nonequivalent) sets of sentences have mathematically indistinct classes of models. But ideally, a semantic theory would also be nonredundant, in the sense that no two distinct semantic structures would represent the same (i.e., logically equivalent) sets of sentences. Standard first-order logic does not measure up to this ideal, because it is unable to distinguish between nonisomorphic structures in general. That is, "elementary equivalence," equivalence as far as first-order theories go, does not coincide with equivalence up to isomorphism, a relation that distinguishes any two nonisomorphic structures.⁶ This inadequacy can be "blamed" either on the

4

excesses of the model-theoretic semantics or on the scantiness (expressive "poverty") of the standard first-order language. Accordingly, we can either make the semantic apparatus less distinctive or strengthen the expressive power of the standard language so that the model-theoretic semantics is put to full use. In any case, it is clear that Tarskian semantics can serve a richer language.

The study of extensions of logic has philosophical, mathematical, and linguistic aspects. Philosophically, my goal has been to find out what distinguishes logical from nonlogical terms, and, on this basis, determine the scope of (core) logic. Once the philosophical question has been decided, the next task is to delineate a complete system of first-order logic, in a sense analogous to that of the expressive completeness of various systems of truth-functional logic. In the early days of modern logic, truthfunctionality was identified as the characteristic property of the "logical" sentential connectives, and this led to the semantics of truth tables and to the correlation of truth-functional connectives with Boolean functions from finite sequences of truth values to a truth value. That in turn enabled logicians to answer the question, What are all the truth-functional sentential connectives? and to determine the completeness (or incompleteness) of various sets of connectives. We cannot achieve the same level of effectiveness in the description of quantifiers. But we can try to characterize the logical quantifiers in a way that will reflect their structure (meaning), show how to "calculate" their value for any set of predicates (relations) in their domain, and describe the totality of quantifiers as the totality of functions of a certain kind. This task takes the form of construction, description, or redefinition, depending on whether the notion of a logical term that emerges out of the philosophical investigations has been realized by an existing formal system.

A further goal is a solid conceptual basis for the generalizations. In his *Introduction to Mathematical Philosophy* (1919) Russell says, "It is a principle, in all formal reasoning, to generalize to the utmost, since we thereby secure that a given process of deduction shall have more widely applicable results."⁷ One of the lessons I have learned in the course of studying extensions of logic is that it is not always clear what the unifying idea behind a given generalized quantifiers, for example, it is not immediately clear what generalization expresses the idea of a *logical* quantifier. Indeed, the need to choose among alternative generalizations has been one of the driving forces behind my work.

New Bounds?

Another angle from which I examine new forms of quantification is that of the ordering of quantifier prefixes: Why should quantifier prefixes be linearly ordered? Are partially-ordered quantifiers compatible with the principles of logical form? In his 1959 paper "Some Remarks on Infinitely Long Formulas," L. Henkin first introduced a new, nonlinear quantifier prefix (with standard quantifiers). Henkin interpreted his new quantifiers, branching or partially-ordered quantifiers, by means of Skolem functions.⁸ An example of a branching quantification is

(1) $(\forall x)(\exists y) \rightarrow \Phi(x, y, z, w),$ $(\forall z)(\exists w) \rightarrow \Phi(x, y, z, w),$

which is interpreted, using Skolem functions, as

(2) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^1(z)].$

However, attempts to extend Henkin's definition to generalized quantifiers came upon great difficulties. Only partial extensions were worked out, and it became clear that the concept of branching requires clarification. This is another case of a generalization in need of elucidation, and conceptual analysis of the branching structure is attempted in chapter 5.

The philosophical outlook underlying this work can be described as follows. Traditionally, logic was thought of as something to be discovered once and for all. Our thought, language, and reasoning may be improved in certain respects, but their logical kernel is fixed. Once the logical kernel is known, it is known for all times: we cannot change—improve or enrich —the logic of our language, reasoning, thought. On this view, questions about the logical structure of human language have definite answers, the same for every language. As the logical structure of human thinking is unraveled, it is encoded in a formal system, and the logical forms of this system are all the logical forms there are, the only logical forms. End of story.

This approach is in essence characteristic of many traditional philosophers, e.g., Kant in *Critique of Pure Reason* (1781/1787) and *Logic* (1800). The enterprise of logic, according to the *Critique*, consists in making an "inventory" of the "formal rules of all thought." These rules are simple, unequivocal, and clearly manifested. There is no questioning their content or their necessity for human thought. Because of the limited nature of its task, logic, according to Kant, "has not been able to advance a single step [since Aristotle], and is thus to all appearances a closed and completed body of doctrine."⁹ That this view of logic is not accidental to

Kant's thought is, I think, evident from the use he makes of it in establishing the Table of Categories. The Table of Categories is based on the Table of the Logical Functions of the Understanding in Judgments, and the absolute certainty regarding the latter provides, according to Kant, an "unshakeable" basis for the former.

I, for one, do not share this view of logic. Even if there are "eternal" logical truths, I cannot see why there should be eternal conceptual (or linguistic) carriers of these truths, why the logical structure of human thought (language) should be "fixed once and for all." I believe that new logical structures can be constructed. Some of the innovations of modern logic appear to me more of the nature of invention than of discovery. Consider, for instance, Frege's construal of number statements. Was this a discovery of the form that, unbeknownst to us, we had always used to express number statements, or was it rather a proposal for a new form that allowed us to express number statements more fruitfully?

The intellectual challenge posed by man-made natural language is, to my mind, not only that of systematic description. As with mathematics or literature, the enterprise of language is first of all that of creating language, and this creative project is (in all three areas) unending. Even in contemporary philosophy of logic, most writers seem to disregard this aspect of language, approaching natural language as a "sacred" traditional institution. But does not the persistent, intensive engagement of these same philosophers with ever new alternative logics point beyond a search for new explanations to a search for new forms?

The view that there is no unique language of logic can also be based on a more conservative approach to human discourse. Defining the field of our investigation to be language as we currently use it, we can invoke the principle of multiformity of language, which is the linguistic counterpart of what H. T. Hodes called Frege's principle of the "polymorphous composition of thought."¹⁰ Consider the following sentences:¹¹

(3) There are exactly four moons of Jupiter.

(4) The number of moons of Jupiter = 4.

It is crucial for Frege, as Hodes emphasizes, that (3) and (4) express the same thought. The two sentences "differ in the way they display the composition of that thought, but according to Frege, one thought is not composed out of a unique set of atomic senses in a unique way."¹² Linguistically, this means that the sentence

(5) Jupiter has 4 moons,¹³

New Bounds?

6

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which can be paraphrased both by (3) and by (4), has both the logical form

(6) (!4x)Ax

and the logical form

(7) $(\iota x)Bx = 4$.

I think this principle is correct. Once we accept the multiformity of language, change in the "official" classification of logical terms is in principle licensed.

The logical positivists, unlike the traditional philosophers, made change in logic possible. Indeed, they made it too easy. Logic, on their view, is nothing more than a linguistic convention, and convention is something to be kept or replaced, at best on pragmatic grounds of efficiency (but also just on whim). I sympathize with Carnap when he says, "This [conventionall view leads to an unprejudiced investigation of the various forms of new logical systems which differ more or less from the customary form ..., and it encourages the construction of further new forms. The task is not to decide which of the different systems is 'the right logic' but to examine their formal properties and the possibilities for their interpretation and application in science."¹⁴ Furthermore, I agree that accepting a new logic is adopting a new linguistic framework and that such "acceptance cannot be judged as being either true or false because it is not an assertion. It can only be judged as being more or less expedient, fruitful, conducive to the aim for which the language is intended."¹⁵ What I cannot agree with is the insistence on the exclusively practical nature of the enterprise: "the introduction of the new ways of speaking does not need any theoretical justification ... to be sure, we have to face at this point an important question; but it is a practical, not a theoretical question."¹⁶

In my view, revision in logic, as in any field of knowledge, should face the "trial of reason" on both fronts, practice and theory. The investigations carried out in this essay concern the theoretical grounds for certain extensions of logic.

Generalized quantifiers have attracted the attention of linguists, and some of the most interesting and stimulating works on the subject come from that field. Quantifiers appear to be the closest formal counterparts of such natural-language determiners as "most," "few," "half," "as many as," etc. This linguistic perspective received its first elaborate and systematic treatment in Barwise and Cooper's 1981 paper "Generalized Quantifiers and Natural Language." Much current work is devoted to continuing Barwise and Cooper's enterprise.¹⁷ The discovery of branching quantifiers

8

New Bounds?

in English is credited to J. Hintikka in "Quantifiers vs. Quantification Theory" (1973). Hintikka's paper aroused a heated discussion and steps towards a systematic linguistic analysis of branching quantifiers were taken by Barwise in "On Branching Quantifiers in English" (1979).

The work on generalized and branching quantifiers in linguistics, though answering high standards of formal rigor, has a strong empirical orientation. As a result, study of the "data" is given precedence over "pure" conceptual analysis. The task of formulating a cohesive empirical theory is particularly difficult in the case of branching quantifiers because evidence is so scarce. In fact, while the branching form appears to be grammatical, it is arguable whether it has, in actual languages, a clear semantic content. To me, this grammatical form appears to be "in search of a content." In any case, my own work emphasizes the conceptual aspect of the branching form. The direction of analysis is from philosophy to logic to natural language. This has the advantage, if the attempt is successful, that the theory is not piecemeal and the applications follow from a general conception. On the other hand, since empirical evidence is not given precedence, the proposals for linguistic applications are presented merely as theoretical hypotheses, and their empirical value is left for the linguist to judge.

My search for new logical forms is prompted by interests on several levels. For one thing, it is a way of asking the general philosophical questions: What is logic? Why should logic take the form of standard mathematical logic? For another, it is an attempt to understand more deeply the fundamental principles of modern logic. Mathematical logic, in particular first-order logic, has acquired a distinguished, paradigmatic place in contemporary analytic philosophy. This situation has naturally led to attempts to extend the range of its applicability, especially to various intensional contexts. It has also led to attacks on the basic principles of the standard system and to the consequent construction of alternative logics. Thus the philosophical scene abounds in modal, inductive, epistemic, deontic, and other extensions of "classical" first-order logic, as well as in intuitionistic, substitutional, free, and other rival logics. However, few in philosophy have suggested that the very principles underlying the "core" first-order logic might not be exhausted by the "standard" version. The present work ventures such a philosophical view, inspired by recent mathematical and linguistic developments. These have not yet received the attention they warrant in philosophical circles, and the opportunity they provide for a reexamination of fundamental principles underlying modern logic has largely passed unnoticed. The realization of

this opportunity motivates my work. Logic, I believe, is a vehicle of thought. This work is done with the hope of contributing to the understanding of its scope.

The Initial Generalization

1 Mostowski and Frege

In the 1957 paper "On a Generalization of Quantifiers," A. Mostowski introduced linguistic operators of a new kind that, he said, "represent a natural generalization of the logical quantifiers."¹ Syntactically, Mostowski's quantifiers are formula building, variable binding operators similar to the existential and universal quantifiers of standard first-order logic. That is, if Φ is a formula, the operation of quantification by a Mostowskian quantifier Q and an individual variable x yields a more complex formula, $(Qx)\Phi$, in which x is bound by Q. Semantically, Mostowski's operators are functions that assign a truth value to any set of elements in the universe of a given model in such a way that the value assigned depends on the cardinalities of the set in question and its complement in the universe and on nothing else. Since the standard existential and universal quantifiers can also be defined in that manner, the new operators constitute a generalization of quantifiers in the semantic sense too. There thus exist, according to Mostowski, a great many operators on formulas with syntactic and semantic features similar to those of the standard quantifiers. These constitute a genuine extension of the logical quantifiers.

To understand Mostowski's generalization more deeply, I will begin with a short regression to Frege. Frege construed the existential and universal quantifiers as second-level quantitative properties that hold (or do not hold) of a first-level property in their range due to the size of its extension. This characterization of quantifiers is brought out most clearly in Frege's analysis of existence as a quantifier property in *The Foundations* of Arithmetic (1884): "Existence is a property of concepts."² "Affirmation of existence is in fact nothing but denial of the number nought."³ "The proposition that there exists no rectangular equilateral rectilinear triangle ... state[s] a property of the concept 'rectangular equilateral rectilinear triangle'; it assigns to it the number nought."⁴

Within Frege's hierarchy of levels a (first-order) quantifier is a 1-place second-level predicate the argument place of which is to be filled by a 1-place first-level predicate (the argument place of which is in turn to be filled by a singular term). A sentence of the form $(\exists x)\Phi x$ is true if and only if (henceforth, "iff") the extension of the 1-place predicate (or propositional function) $\Phi \xi$ is of cardinality larger than 0. And a sentence of the form $(\forall x)\Phi x$ is true iff the extension of $\Phi \xi$ is the whole universe, or its counterextension has cardinality 0.⁵

This Fregean conception of the standard quantifiers underlies Mostowski's generalization. In Mostowski's model-theoretic terminology, the standard quantifiers are interpreted as functions on sets (universes of models) as follows:

(1) The universal quantifier is a function \forall such that given a set A, $\forall(A)$ is itself a function $f: P(A) \rightarrow \{T, F\}$, where P(A) is the power set of A and for any subset B of A,

$$f(B) = \begin{cases} T & \text{if } |A - B| = 0\\ F & \text{otherwise.} \end{cases}$$

(2) The existential quantifier is a function \exists such that given a set A, $\exists (A)$ is a function $g : P(A) \rightarrow \{T, F\}$, where for any subset B of A,

$$g(B) = \begin{cases} T & \text{if } |B| > 0 \\ F & \text{otherwise} \end{cases}$$

On the basis of this definition we can associate with each quantifier Q(\forall or \exists) a function t^{Q} that tells us, given the size of a universe A, under what numerical conditions Q gives a subset B of A the value "true." Thus, the function t^{Q} may be defined as a function on cardinal numbers (sizes of universes) assigning to each cardinal number α another function t_{α}^{Q} that says how many objects are allowed to fall under a set B and its complement in a universe of size α in order for Q(B) to be "true." Since the "cardinality image" of each set in a universe of size α can be encoded by a pair of cardinal numbers $\langle \beta, \gamma \rangle$, where β represents the size of B and γ the size of its complement in the given universe, t_{α}^{Q} is defined as a function from all pairs of cardinal numbers β and γ , the sum of which is α , to {T, F}. So the universal quantifier function, t^{\forall} , is defined, for each α , by t_{α}^{\forall} , which assigns to any given pair $\langle \beta, \gamma \rangle$ in its domain a value according to the rule

(3)
$$t_{\alpha}^{\forall}(\beta, \gamma) = \begin{cases} T & \text{if } \gamma = 0 \\ F & \text{otherwise.} \end{cases}$$

12

(4)
$$t_{\alpha}^{3}(\beta, \gamma) = \begin{cases} T & \text{if } \beta > 0 \\ F & \text{otherwise.} \end{cases}$$

We can now define the standard quantifiers in terms of their *t*-functions as follows: Given a set A and a subset B of A,

(5)
$$\forall_{\mathcal{A}}(B) = \begin{cases} T & \text{if } t^{\forall}_{|\mathcal{A}|}(|B|, |\mathcal{A} - B|) = T \\ F & \text{otherwise.} \end{cases}$$

Similarly,

(6)
$$\exists_{A}(B) = \begin{cases} T & \text{if } t^{3}_{|A|}(|B|, |A - B|) = T \\ F & \text{otherwise.} \end{cases}$$

However, \forall and \exists are not the only quantifiers that can be defined by cardinality functions like those above. Any function *t* that assigns to each cardinal number α a function t_{α} from pairs of cardinal numbers $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$ to {T, F} defines a quantifier. Given a set *A* and a subset *B* of *A*, this quantifier is defined on *A* exactly as \forall and \exists are. For example, suppose that the cardinality function t^{δ} is defined, for any cardinal number α and pair $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$, by

(7)
$$t_{\alpha}^{\delta}(\beta, \gamma) = \begin{cases} T & \text{if } \beta = \delta \\ F & \text{otherwise.} \end{cases}$$

Then t^{δ} determines the cardinal quantifier $(!\delta x)$: "for exactly δ elements x in the universe." Similar functions define the quantifiers "for at least δ elements x in the universe" and "for at most δ elements x in the universe." Cardinality statements in general, " δ things have property P," can thus be formalized as first-order quantifications

(8) $(!\delta x) Px$,

which assert that the extension of P has δ elements. In Frege's conceptual scheme, (8) would be a second-level statement that assigns a second-level numerical property to the extension of the first-level predicate P. But this is exactly Frege's own analysis of statements of number: "The content of a statement of number is an assertion about a concept.... If I say 'Venus has 0 moons'... what happens is that a property is assigned to the *concept* 'moon of Venus,' namely that of including nothing under it. If I say 'the King's carriage is drawn by four horses,' then I assign the number four to the concept 'horse that draws the King's carriage.'"⁶ We see that Mostow-ski's generalization is indeed in the spirit of Frege.

Yet numerical quantifiers (finite and infinite) do not exhaust Mostowski's definition. Consider the function t defined (relative to a cardinal The Initial Generalization

number
$$\alpha$$
 and any pair $\langle \beta, \gamma \rangle$ such that $\beta + \gamma = \alpha$) as

(9) $t_{\alpha}(\beta, \gamma) = \begin{cases} T & \text{if } \beta > \gamma \\ F & \text{otherwise.} \end{cases}$

This function defines the quantifier (Mx), "most of the objects in the universe are such that ..." (where we take "Most things are B" to mean "There are more Bs than non-Bs"). Consider also

(10)
$$t_{\alpha}(\beta, \gamma) = \begin{cases} T & \text{if } \beta \text{ is a finite even number} \\ F & \text{otherwise.} \end{cases}$$

This function defines the quantifier (Ex): "an even number of objects in the universe are such that" Another quantifier is defined

(11)
$$t_{\alpha}(\beta, \gamma) = \begin{cases} T & \text{if } \beta = \alpha \\ F & \text{otherwise.} \end{cases}$$

This is the *Chang* or *equicardinal quantifier*: "as many objects as there are elements in the universe are such that"⁷ And so on.

Among the totality of cardinality functions t are functions that assign to different cardinals α different functions t_{α} . Such a "vacillating" function t might be defined for two distinct cardinal numbers α_1 and α_2 by

(12)
$$t_{\alpha_1}(\beta, \gamma) = \begin{cases} T & \text{if } \beta = m \\ F & \text{otherwise,} \end{cases}$$

(13) $t_{\alpha_2}(\beta, \gamma) = \begin{cases} T & \text{if } \beta = n \\ F & \text{otherwise,} \end{cases}$

where $m \neq n$. The function t expresses cardinality properties of sets relative to the size of the universe: "m out of α_1 , n out of α_2 ," Some vacillating functions are reducible to "simple" functions like the ratio function "1/2," which is fixed for all universes. (Thus, 1/2 = 1 out of 2, 2 out of 4, 3 out of 6, etc., where some conventional rule is given for universes with an odd number of elements.) Other vacillating functions represent irregular ratios ("2 out of 3, 3 out of 6, 19 out of 19, ..."), and these are genuinely "manifold" cardinality functions.

According to Mostowski, any formula-binding operator defined by some cardinality function (simple or vacillating) as described above is a generalized quantifier.

2 A Criterion For Logical Quantifiers

Are Mostowski's quantifiers *logical* quantifiers? Are they *all* the logical quantifiers? From a Fregean point of view, standard first-order logic is a

first-level system with one or two 1-place second-level predicates: the existential and/or universal quantifiers. Mostowski's logic is, from this point of view, a first-level system with an arbitrary number of 1-place second-level predicates of the same type as the standard quantifiers (i.e., 1-place second-level predicates of 1-place first-level predicates). However, not all second-level predicates of that type are logical quantifiers, according to Mostowski's definition. The predicate "P is a (first-level) attribute of Napoleon" is not. More generally, all noncardinality predicates are excluded from this category. The question naturally arises as to why the distinction between logical and nonlogical predicates should coincide with the distinction between cardinality properties and noncardinality properties. What does cardinality have to do with logicality?

Mostowski's answer is that there are two natural conditions on logical quantifiers:

CONDITION LQ1 "Quantifiers enable us to construct propositions from propositional functions,"8

CONDITION LQ2 A logical quantifier "does not allow us to distinguish between different elements of [the universe]."9

The first requirement is clear. Syntactically, a quantifier is a formulabuilding expression that operates by binding a free variable in the formula to which it is attached and thus in finitely many applications generates a sentence, i.e., a closed formula.

The second, semantic requirement Mostowski interprets as follows: a l-place first-level propositional function Φx satisfies a quantifier Q in a given model 21 only if any 1-place first-level propositional function whose extension in \mathfrak{A} can be obtained from that of Φx by some permutation of the universe satisfies it as well. More succinctly, logical quantifiers are invariant under permutations of the universe in a given model for the

It is interesting to note that (LQ2) is also suggested by Dummett in Frege: Philosophy of Language (1973):

Let us call a second-level condition any condition which, for some domain of objects, is defined, as being satisfied or otherwise, by every predicate which is in turn defined over that domain of objects. Among such second-level conditions, we may call a quantifier condition any which is invariant under each permutation of the domain of objects: i.e. for any predicate 'F(ξ)' and any permutation φ , it satisfies 'F(ξ)' just in case it satisfies that predicate which applies to just those objects $\varphi(a)$, where 'F(ξ)' is true of a. Then we allow as also being a logical

The Initial Generalization

14

constant any expression which ... allows us to express a quantifier condition which could not be expressed by means of ... [the universal and existential] quantifiers and the sentential operators alone.¹⁰

Now it is a metatheoretical fact about first-order models that given a model \mathfrak{A} with a universe A and a 1-place second-level property $\mathfrak{P}, \mathfrak{P}$ satisfies (LO2) with respect to the elements of A iff \mathcal{P} is a cardinality property. And this explains why Mostowski identifies logical quantifiers with cardinality quantifiers. (A theorem establishing the one-to-one correspondence between quantifiers satisfying (LQ2) and cardinality quantifiers was proved by Mostowski. See the appendix.)

To sum up, syntactically, a quantifier is an operator binding a formula by means of an individual variable. Semantically, it is a function that assigns to every universe A an A-quantifier (or a quantifier on A), Q₄, Q₄ is itself a function from subsets of A into $\{T, F\}$. We will call the cardinality function t associated with a given A-quantifier Q, the cardinality counter*part* of O_{4} and symbolize it by t_{2}^{q} (or sometimes simply by t). Quantifiers satisfying (LQ1) and (LQ2) are Mostowskian quantifiers. More precisely, a Mostowskian quantifier is a quantifier Q satisfying (LQ1) and such that for every set A, Q_A satisfies (LQ2), and if A_1 , A_2 are sets of the same cardinality, then Q_{A_1} and Q_{A_2} have the same cardinality counterpart. For exact definitions, see the appendix.

It is worthwhile to note that Mostowski's system of generalized quantifiers exhausts the 1-place second-level predicates that satisfy (LQ2) only relative to the standard semantics for first-order logic. Disregarding the particular features of this semantics, we can say that any second-level predicate embodying some measure of sets and insensitive to the identity of their members satisfies this condition. Mostowski's quantifiers express measures of a particular kind, namely measures that have to do with the cardinality of sets, and as we have seen, these are all the second-level 1-place "measure predicates" satisfying (LQ2) relative to standard model theory. But these are not the only second-level measures conforming to (LQ2). Other quantifier measures of first-level extensions have been developed involving more elaborate model structures. Barwise and Cooper (1981) describe two such cases. The first is a quantifier Q, studied by Sgro (1977), where " $(Qx)\Phi x$ " says that the extension of Φx contains a nonempty open set. This quantifier requires that models be enriched by some measure of distance (topology). The second has to do with measures of infinite sets: "Measures have been developed in which (a) and (b) make perfectly good sense.

15

2

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3

(a) More than half the integers are not prime.

(b) More than half the real numbers between 0 and 1, expressed in decimal notation, do not begin with 7."¹¹

Under the same category fall probability quantifiers, defined over nonstandard models in which probability values are assigned to extensions of predicates.¹² We will not take up quantifiers for nonstandard models here. We will also limit ourselves to finitistic logics. This is because the extension to infinitely long formulas is not necessary to investigate the generalized notion of a logical term, which is what interests us in this work.¹³

To return to Mostowski, the syntax of a first-order logic with (a finite set of) Mostowski's generalized quantifiers is the same as the syntax of standard first-order logic with two exceptions: (1) The language includes finitely many quantifier symbols, Q_1, Q_2, \ldots, Q_n (among them possibly, but not necessarily, \forall and \exists). (2) The rule for building well-formed quantified formulas is, If Φ is a well-formed formula, then $(Q_1x)\Phi, (Q_2x)\Phi, \ldots,$ $(Q_nx)\Phi$ are all well-formed formulas (for any individual variable x).

We can extend the Tarskian definition of satisfaction to cardinality quantifiers by replacing the entry for quantified formulas by the following. Let \mathfrak{A} be a (standard) first-order model, and let A be the universe of \mathfrak{A} . Let g be an assignment of members of A to the variables of the language.

• If Φ is a formula and Q a quantifier symbol, $\mathfrak{A} \models (Qx)\Phi[g]$ iff for some α and β such that $\alpha + \beta = |A|$ and $t_A^Q(\alpha, \beta) = T$, there are exactly α elements $a \in A$ such that $\mathfrak{A} \models \Phi[g(x/a)]$ and exactly β elements $b \in A$ such that $\mathfrak{A} \models \sim \Phi[g(x/b)]$,

where " $\mathfrak{A} \models \Phi[g]$ " is to be read, " Φ is satisfied in \mathfrak{A} by g" and g(x/a) is an assignment of members of A to the variables of the language that assigns a to x and otherwise is the same as g. Informally, the definition now says that $(Qx)\Phi x$ is true in \mathfrak{A} iff the number of elements in A satisfying Φx and the number of elements in A not satisfying Φx are as t_A^Q allows. Note the following:

• The definition of satisfaction above is a schema that, for any given quantifier, instantiates differently. In the cases of \forall and \exists the schema instantiates in the standard way. In the case of the quantifier "most" the definition is, if Φ is a formula, then $(Mx)\Phi x$ is true in \mathfrak{A} iff the number of Φx 's in \mathfrak{A} is larger than the number of non- Φx 's in \mathfrak{A} . (Formally, $\mathfrak{A} \models (Mx)\Phi[g]$ iff for some α , β such that $\alpha + \beta = |A|$ and $\alpha > \beta$, there are exactly α elements a in A such that $\mathfrak{A} \models \Phi[g(x/a)]$ and exactly β elements 5 in A such that $\mathfrak{A} \models \sim \Phi[g(x/b)]$.)

The Initial Generalization

16

• The definition of a model for first-order logic with Mostowskian generalized quantifiers is the same as that for standard first-order logic. A model for the extended logic does not contain any new "entities" not found in models for standard logic. The only difference is in the computation of truth values for quantified formulas, given a model and an assignment of objects in the universe to the variables of the language.

An important difference between Mostowski's system and standard first-order logic is that the former is not in general complete. Thus, for example, Mostowski proved that if the quantifiers of a generalized firstorder logic include the existential or universal quantifier and at least one quantifier Q satisfying condition (*) below, then the logic is incomplete: not all logically true sentences of this system are provable, or, the set of sentences true in all models for the logic is not recursively enumerable.

CONDITION (*) The cardinality function t^Q associated with Q assigns to \aleph_0 a function $t^Q_{\aleph_0}$ such that both $\{n : t^Q_{\aleph_0}(n, \aleph_0) = F\}$ and $\{m : t^Q_{\aleph_0}(m, \aleph_0) = T\}$ are denumerable.¹⁴

An example of a quantifier satisfying condition (*) is Q, where, given a denumerable universe, " $(Qx)\Phi x$ " means "The number of Φx 's is even."

On the other hand some generalized logics are complete. For example, H. J. Keisler (1970) proved that the logic obtained from standard firstorder logic with identity by adding the quantifier "there are uncountably many" and a modest set of axiom schemas is complete.

Referring to the incompleteness of first-order logic with generalized quantifiers in general, Mostowski says, "In spite of this negative result we believe that some at least of the generalized quantifiers deserve a closer study and some deserve even to be included into systematic expositions of symbolic logic. This belief is based on the conviction that the construction of formal calculi is not the unique and even not the most important goal of symbolic logic."¹⁵

One goal for which completeness appears to be immaterial is the characterization of the logical structure of natural language.

3 Generalized Quantifiers and Natural Language

In their seminal paper "Generalized Quantifiers and Natural Language" J. Barwise and R. Cooper examined Mostowski's theory from a linguistic perspective. Mostowski's logic, Barwise and Cooper observe, is superior

18

to standard first-order logic in its account of natural-language quantification. "The quantifiers of standard first-order logic are inadequate for treating the quantified sentences of natural languages" in part because "there are sentences which simply cannot be symbolized in a logic which is restricted to ... \forall and \exists ."¹⁶ Mostowski's method, on the other hand, allows us to encode the structure of such sentences as defy the standard analysis. Let me give a few examples:

(14) There are only a finite number of stars.

(15) No one's heart will beat an infinite number of times.¹⁷

(16) There is an even number of letters in the English alphabet.

(17) The number of rows in a (full) truth table is a power of 2.

(18) There are 2^{\aleph_0} reals between any two nonidentical integers.

The formal structure of (14) to (18) is analyzed in Mostowski's logic as follows:

(19) (Finite x) star x,

where for each model \mathfrak{A} with universe A, $t_A^{\text{finite}}(\alpha, \beta) = T$ iff $\alpha < \aleph_0$;

(20) $\sim (\exists x) (\exists y) [y \text{ is a heart of } x \& (\inf z) z \text{ is a beat of } y],$

where $t_A^{\inf}(\alpha, \beta) = T$ iff $\alpha \ge \aleph_0$;

(21) (Even x) x is a letter in the English alphabet,

where t_A^{even} is defined by (10) above;

(22) $(\forall y)[y \text{ is a (full) truth table} \rightarrow (\text{Power-of-2 } x) x \text{ is a row in } y],$

where $t_A^{\text{power-of-2}}(\alpha, \beta) = T$ iff α is a power of 2;

(23) $(\forall x)(\forall y)[x \text{ is an integer } \& y \text{ is an integer } \& x \neq y \rightarrow (2^{\aleph_0}z)(z \text{ is a real number } \& z \text{ is between } x \text{ and } y)],$

where $t_A^{2^{\aleph_0}}(\alpha, \beta) = T$ iff $\alpha = 2^{\aleph_0}$. What about (24) to (29)?

what about (24) to (29)?

- (24) More than one third of the population of the world suffers from hunger.
- (25) 94% of all Americans believe in God.
- (26) Some recipients of a Nobel prize are known to most people in the world.
- (27) Most people are not hostile to most people.
- (28) As many Israelis are liberals as are not.
- (29) \aleph_0 natural numbers are prime, and the same number are not prime.

The Initial Generalization

Do these resist a first-order symbolization? No, say Barwise and Cooper. By inserting nonlogical, "domain-fixing" axioms of the form $(\forall x)\Phi x$, by introducing many-sorted variables, or by limiting consideration to particular models, one can use Mostowski's quantifiers to analyze naturallanguage sentences like (24) to (29). Thus we might formalize (24) to (29) as follows:

(30) (More-than-1/3 x) x is suffering from hunger,

where $t_A^{\text{more-than-1/3}}(\alpha, \beta) = T$ iff $\alpha > 1/3(\alpha + \beta)$, both α and β are finite, and the range of x is the set of all people in the world (at the present, say).

(31) (94% x) x believes in God,

where $t_A^{94\%}(\alpha, \beta) = T$ iff $\alpha = 94\%$ $(\alpha + \beta)$, α, β are finite, and the range of x is the set of all American people (at the present).

(32) $(\exists x)[x \text{ is a recipient of a Nobel prize & (Most y) x is known to y]}$

Here t_A^{most} is defined by (9) above. The range of x is as in (30).

(33) a. \sim (Most x)(Most y) x is hostile to y

b. (Most x) ~ (Most y) x is hostile to y

c. (Most x)(Most y) ~ (x is hostile to y)

whether the analysis is (a), (b), or (c) depends on how you read the negation in (27). Both t_A^{most} and the range of x are as above.¹⁸

(34) (As-many-as-not x) x is a liberal,

where $t_A^{\text{as-many-as-not}}(\alpha, \beta) = T$ iff $\alpha \ge \beta$, and the range of x is the set of Israelis.

(35) $(\aleph_0/\aleph_0 x) x$ is prime,

where $t_A^{\aleph_0/\aleph_0}(\alpha, \beta) = T$ iff $\alpha = \beta = \aleph_0$, and the range of x is the set of natural numbers.

Clearly, the natural-language "most," "almost all," "few," "a few," "many," etc. can be construed as Mostowskian quantifiers only to the extent that they can be given absolute cardinality values (or ranges of values). Under such a construal, we read "most" as "(cardinalitywise) more than a half," just as in standard logic we read "some" as "at least one."

What are the limitations of Mostowski's system from the point of view of the logical structure of natural language? Consider the following sentences:

(36) Most of John's arrows hit the target.¹⁹

(37) 60% of the female students in my class are A-students.

The Initial Generalization

20

- (38) The majority of children who do not communicate with anyone during the first two years of their lives are autistic.
- (39) Most of the students in most colleges are not exempt from tuition fees.

Delimiting the range of the bound variables, which enabled us to analyze (24) to (29), is inadequate for the formalization of (36) to (39). Restricted or sorted domains are useful up to the point where they become utterly artificial, as they would if they were used to analyze (36) to (39) with Mostowski's quantifiers. Mostowski's system is rich enough to analyze sentences of the form

(40) Such and such a quantity of all the objects that there are, are B,

but in general is inadequate for the analysis of sentences of the form

(41) Such and such a quantity of all As are Bs.

This verdict was reached both by Barwise and Cooper and by N. Rescher in "Plurality-Quantification" (1962) and elsewhere.²⁰ Clearly, statements of type (41) cannot be symbolized as

(42) $(Qx)(Ax \rightarrow Bx)$,

as can be seen by the following counterexample: Suppose that only one third of the things that satisfy Ax in a given model \mathfrak{A} satisfy Bx in \mathfrak{A} . Suppose also that most of the things in the universe of \mathfrak{A} do not satisfy Ax. Then $(Mx)(Ax \to Bx)$ will come out true in \mathfrak{A} (most things in \mathfrak{A} satisfy " $Ax \to Bx$ " by falsifying the antecedent), although it is plainly false that most of the As in \mathfrak{A} are Bs.

In general, a statement of the form (41) cannot be formalized by a formula of the form

(43) $(Qx)\Phi x$.

(A theorem to the effect that "more than half of the As" cannot be defined in terms of "more than half of all things that there are" using the apparatus of standard first-order logic was proved by Barwise and Cooper for finite universes and by D. Kaplan for the infinite case.²¹) Rescher concludes.

Textbooks often charge that traditional logic is "inadequate" because it cannot accommodate patently valid arguments like (1) [All A's are B's \vdash All parts of A's are parts of B's]. But this holds equally true of modern quantificational logic itself, which cannot accommodate (2) [Most things are A's; Most things are B's \vdash Some A's are B's] until supplemented by something like our plurality-quantification [Mostowski's "most"]. And even such expanded machinery cannot accommodate (3) [Most C's are A's; Most C's are B's \vdash Some A's are B's]. Powerful tool though it is, quantificational logic is unequal to certain childishly simple valid arguments.²² Barwise and Cooper's strategy in the face of the alleged inexpediency of quantificational logic is to give up logical quantifiers altogether. The idea underlying this move seems to be the following: There is no absolute meaning to such expressions as "more than half." The quantities involved in "more than half the natural numbers between 0 and 10" are different from those involved in "more than half the natural numbers between 0 and 100." Hence "more than half" cannot be interpreted independently of the interpretation of the set expression attached to it. Thus in the schema

(44) More than half the As are Bs,

"'more than half' is not acting like a quantifier, but like a determiner. It combines with a set expression to produce a quantifier."²³ "Quantifiers correspond to noun-phrases, not to determiners."²⁴ The quantifier in (44) is the whole noun phrase "more than half the As," and (44) is rendered

(45) (More-than-half-A x)Bx.²⁵

In this way the indeterminacy inherent in determiners is resolved by the set expressions attached to them, and the difficulty indicated above disappears: "more than half the natural numbers between 0 and 10" and "more than half the natural numbers between 0 and 100" are two distinct quantifiers, each with its own meaning. And in general, quantifiers are pairs, $\langle D, S \rangle$, of a determiner D and a set expression S. If S, S' denote different sets, DS and DS' are different quantifiers.

What about "every" and "some"? According to Barwise and Cooper, the situation is indeed different in the case of "every." The schema

(46) Every A is B

can be expressed in terms of the quantifier "every" independently of the interpretation of A:

(47) (Every x)($Ax \rightarrow Bx$).

However, they say, the syntactic dissimilarity of (46) and (47) indicates that even in this case the "true" quantifier is "every A." "Every" is but a determiner, although, unlike "more than half," it is a logical determiner. Sentence (46), then, is to be symbolized not as (47) but rather as

(48) (Every-A x)Bx.

4 Nonlogical Quantifiers

As a theory of quantification, Barwise and Cooper's theory is evidently very bloated. "Every man," "every woman," "every child," "every son of mine," etc. are all different quantifiers. So are "most men," "most

The Initial Generalization

22

women," "most children," and so on. Two questions present themselves: Is such an excessive theory of quantifiers necessary to account for the diverse patterns of quantification in natural-language discourse? Is what this theory explains quantification?

Barwise and Cooper address the following questions:

• What is the role of quantifiers and how are they interpreted in a model? According to Barwise and Cooper, we use quantifiers to attribute properties to sets. " $\exists x\varphi(x)$ " asserts that the set of things satisfying $\varphi(x)$ is not empty. " $\forall x\varphi(x)$ " says that the set of φ s contains all the objects in the universe of discourse. "Finite $x\varphi(x)$ " states that this set is finite. And so on. Model-theoretically, a quantifier partitions the "family" of all subsets of the universe of a given model into those that satisfy it and those that do not. When combined with the former, it yields the value T, when combined with the latter, the value F. Thus a quantifier can be identified with the family of all sets to which it gives the value T. (Note that according to this account *all* properties of sets are quantifier properties.)

• What is the syntactic category of the natural-language expressions that function as quantifiers? Barwise and Cooper observe that noun phrases in general behave like quantifiers. Given a noun phrase, some verb phrases will combine with it to produce true sentences, and others to produce false sentences. Semantically, this means that each noun phrase divides the family of verb-phrase denotations in a given model into two groups: those that satisfy it and those that do not. Therefore, Barwise and Cooper conclude, "the noun phrases of a language are all and only the quantifiers over the domain of discourse."26 To make their treatment of noun phrases uniform, Barwise and Cooper have to show that proper names can also be treated as quantifiers. But this is not difficult to show. We can treat a proper name like "Harry" as partitioning all the sets in the universe into those that contain Harry and those that do not. Thus "Harry" can be semantically identified with the family of all sets that include Harry as a member. "In our logic," Barwise and Cooper say, "(a) may be translated as (b), or rather, something like (b) in structure.

(a) Harry knew he had a cold.

(b) Harry $\hat{x}[x \text{ knew } x \text{ had a cold}]$."²⁷

In sum, "Proper names and other noun-phrases are natural language quantifiers."²⁸

The linguistic logic developed by Barwise and Cooper and based on the above principles differs from standard first-order logic and its Mostowskian extension in several substantive ways. We can outline its main features as follows:

Syntactically, the logic excludes logical quantifiers altogether. Instead, it includes logical and nonlogical determiner symbols (the logical determiners include "some," "every," "no," "both," "1" (in the sense of "at least 1"), "2," "3," ..., "11" ("exactly one"), "12," ..., "the 1," ...; the nonlogical determiners include "most," "many," "few," "a few," ...). Quantifiers are nonlogical complex terms representing noun phrases in general: "John," "Jerusalem," "most people," "five boys," etc. Formally, quantifiers are defined terms of the form $D(\eta)$, where D is a determiner and η is a (first-level) predicate with a marked argument place called a "set term." A quantified formula is of the form $Q(\eta)$, Q being a quantifier and η a set term. The language also includes the distinguished 1-place first-level predicate (set term) "thing."

Semantically, a model & for the logic provides, in addition to the standard universe of objects, interpretations I for the truth-functional connectives, "thing," and the logical as well as nonlogical determiners. "Thing" is interpreted as the universe of the model, E; each (logical or nonlogical) determiner is interpreted as a function that assigns to every set in the model a family of sets in the model. $\Im(!n) = !n : P(E) \rightarrow P(P(E))$ such that for each $A \subseteq E$, $\ln(A) = \{X \subseteq E : |A \cap X| = n\}$; $\Im(Most) = most$: $P(E) \rightarrow P(P(E))$ such that for each $A \subseteq E$, most $(A) = \{X \subseteq E : |A \cap X| > A \in A\}$ |A - X|; etc. The truth-functional connectives are interpreted in the usual way (although Barwise and Cooper favor a trivalent logic to allow for determiners denoting partial functions). Quantifiers (nonlogical terms) -D(n) for some determiner D and set term n-are interpreted in each model as the family of sets assigned in this model by the denotation of Dto the denotation of η . For example, $\Im(\ln(\operatorname{man})) = \{X \subseteq E : | \{x : x \text{ is a } \}$ man $\{\cap X \mid = n\}$ and $\Im(John) = \{X \subseteq E : John \in X\}$. If Φ and Ψ are 1place predicate symbols (set terms), " $(D\Phi)[\Psi]$ " is true in \mathfrak{E} iff the denotation of Ψ in \mathfrak{E} is a member of the family of sets assigned to $D\Phi$ in \mathfrak{E} .

Barwise and Cooper posit a universal semantic constraint on naturallanguage determiners: "It is a universal semantic feature of determiners that they assign to any set A a quantifier (i.e. family of sets) that *lives on* A,"where Q *lives on* A iff for any set X, $X \in Q$ iff $A \cap X \in Q$.²⁹ The following equivalences illustrate this notion:

Many men run \leftrightarrow Many men are men who run Few women sneeze \leftrightarrow Few women are women who sneeze John loves Mary \leftrightarrow John is John and loves Mary ³⁰ "The quantifiers represented by the subjects of the sentences," Barwise and Cooper explain, "live on the set of men, women and the singleton set containing John, respectively."³¹ And they conclude, "When we turn to non-logical determiners, [the living on constraint] is the only condition we impose as part of the *logic*."³² This condition on determiners is ipso facto a condition on quantifiers.

Is what Barwise and Cooper's theory explains quantification? To resolve this issue, let us consider several intermediate questions: In what sense is Barwise and Cooper's logic a first-order system, given that quantifiers are nonlogical second-level predicates? Quantifiers, in Barwise and Cooper's system, are 1-place predicates. Is being 1-placed an essential property of quantifiers? If so, why? How does Barwise and Cooper's criterion for "quantifierhood" compare with Mostowski's? Does their theory account for all natural-language quantifiers intuitively satisfying Mostowski's principles? Does it account only for such quantifiers?

Obviously, Barwise and Cooper's requirements on quantifiers are altogether different from those of Mostowski (and Dummett). In particular, Barwise and Cooper's quantifiers do not satisfy the semantic condition (LQ2). These quantifiers do in general distinguish elements in the universe of a model for their system. Consider the following two pairs of sentences:

- (49) a. Einstein $\hat{x}[x \text{ is among the ten greatest physicists of all time]}$
 - b. Einstein $\hat{x}[x \text{ is among the ten greatest novelists of all time]}$
- (50) a. Most (natural numbers between 1 and 10) \hat{x} [x < 7]
 - b. Most (natural numbers between 1 and 10) \hat{x} [9 < x < 17].

Although the extension of "x is among the ten greatest physicists of all time" can be obtained from that of "x is among the ten greatest novelists of all time" by a permutation of the universe of discourse, the quantifier "Einstein" assigns the two sets different truth values. Similarly, "Most natural numbers between 1 and 10" assigns different truth values to the extensions of "x < 7" and "9 < x < 17" in spite of the fact that the one extension can be obtained from the other by some permutation of the (intended) universe.

Moreover, not all quantity properties, properties that satisfy Mostowski's criterion, are quantifiers (or constituents of quantifiers, i.e., determiners) on Barwise and Cooper's view. Thus the requirement that quantifiers "live on" the sets in their domain excludes some linguistic constructions that we would expect to be analyzed by means of cardinality quantifiers:

(51) Mostly women have been elected to Congress.

The Initial Generalization

(52) Only human beings have brains.

(53) Not only men are allowed in the club.

Sentence (51) cannot be formalized as

(54) Mostly (women) \hat{x} [x has been elected to Congress],

since (54), unlike (51), comes out true in the intended model: the requirement of "living on" implies that for any set X, "Mostly (women)" is true of X if it is true of $\{x : x \text{ is a woman}\} \cap X$. But clearly, "Mostly women are women elected to Congress" is true. Similar problems occur if we want to formalize (52) as

(55) Only (human beings) \hat{x} [x has a brain],

or (53) as

(56) Not only (men) \hat{x} [x is allowed in the club].

The requirement of "living on" determines that (55) is true and (56) is false, while (52) is clearly false, and (53) is sometimes true. Barwise and Cooper can get over the difficulty by rephrasing (51) to (53) with determiners other than "mostly," "only," and "not only," respectively, but only at the expense of giving up the desideratum of transparent analysis of linguistic structures.

The requirement that only *noun phrases* be construed as quantifiers also blocks "natural" candidates for natural-language quantifiers. Consider the following sentences involving quantitative comparison of extensions, which, by Dummett's and Mostowski's criterion, have a good claim to being analyzed as statements of quantification:

- (57) There are fewer men than women.
- (58) More people die of heart disease than die of cancer.
- (59) They are outnumbered by us.
- (60) The same percentage of boys and girls who took the test received a perfect score.

Clearly, the operation of quantification in (57) to (60) is not carried out by noun phrases.

To sum up, Barwise and Cooper's theory is clearly not based on Mostowski's ideas of the nature of quantifiers. In particular, Mostowski's semantic condition, (LQ2), is violated by Barwise and Cooper. Their theory also does not offer an alternative principle to (LQ2) of the same general import as Mostowski's. In my opinion, Barwise and Cooper's analysis explains some linguistic regularities, but what it explains is not the

26

structure of *quantifiers*. Their account is at once too particular to explain the notion of quantification in all its generality—witness (51) to (53) and (57) to (60)—and too general to focus on the unique features of quantifiers —see (49) and (50).³³

5 Logical Quantifiers

Can we increase the expressive power of Mostowski's logical system so that it is no longer subject to Barwise and Cooper's criticism, without betraying its underlying principles?

I think the "inadequacy" of Mostowski's system can be analyzed along lines different from those taken by Barwise and Cooper. The problem is neither with the "logicality" of Mostowski's quantifiers nor with his criterion for second-level predicates expressing quantifier properties (a criterion shared, as we have seen, by Dummett). The problem is that Mostowski explicitly considered only 1-place second-level predicates as candidates for quantifiers. Progress in logic was made after the indispensability of relations was acknowledged. Frege's revolution was in part in recognizing relations for what they are: irreducibly many-place predicates. Mostowski's requirements on quantifiers-that they turn propositional functions into propositions and that they do not distinguish elements in a model for the language-contain nothing to exclude many-place second-level predicates from being first-order quantifiers. On the contrary, the failure of Mostowski's theory to display the quantificational structure of sentences such as (36) to (39) is testimony only to the "incompleteness" of that theory. Mostowski's theory of cardinality quantifiers includes all the "predicative" quantifiers that express cardinality measures, but none of the "relational" quantifiers that express such measures. And there is no reason to believe that Mostowski would have rejected many-place quantifiers.

With this observation the solution to Barwise and Cooper's problem becomes very simple: Both "most" in "most things are A" and "most" in "most As are Bs" are quantifiers, although, as was proved by Barwise and Cooper, the second is not reducible to the first. The first is a 1-place quantifier, M^1 , i.e., a property of first-level properties (or a 1-place function from first-level properties to truth values). It appears in formulas of the form

 $(M^1x)\Phi x$,

and for any given model \mathfrak{A} with universe A it is defined by the function $t_A^{M^1}$ as in (9) above. That is, for all pairs of cardinals α , β whose sum is |A|,

The Initial Generalization

$\iota_A^{\mathsf{M}^1}(\alpha,\beta) = \mathrm{T} \text{ iff } \alpha > \beta.$

The second "most" is a 2-place quantifier, M^2 , i.e., a 2-place *relation* between first-level properties (or a 2-place function from pairs of first-level properties to truth values). It appears in formulas of the form

$(M^2x)(\Phi x, \Psi x),$

and for any given model \mathfrak{A} with universe A it is definable by a function $t_A^{M^2}$, which I will presently characterize.

What kind of relation is M^2 ? It is a relation that in any given universe A holds between two subsets B, B' of A (in that order) iff $|B \cap B'| > |B - B'|$. M_A^2 is defined by the function t from quadruples of cardinals α , β , γ , δ such that $\alpha + \beta + \gamma + \delta = |A|$ to {T, F} as follows:

$\iota_A^{M^2}(\alpha,\beta,\gamma,\delta)=\mathrm{T}\,\mathrm{iff}\,\alpha>\beta,$

where, intuitively, $\alpha = |B \cap B'|$, $\beta = |B - B'|$, $\gamma = |B' - B|$, and $\delta = |A - (B \cup B')|$. Using a Venn diagram, we can form a visual image of the relation between the cardinal numbers α , β , γ , δ and parts of the universe A as in figure 2.1.

It is easy to see that whereas M^2 is not definable in terms of M^1 , M^1 is definable in terms of M^2 . Thus,

 $(M^{1}x)\Phi x$ iff $(M^{2}x)(x = x, \Phi x)$.

Our analysis provides a rationale for extending Mostowski's original system in a way that was first proposed by Per Lindström in "First order Predicate Logic with Generalized Quantifiers" (1966). This extension has been widely adopted by logicians and mathematicians, including Barwise in his purely logical writings. (Barwise later also expressed misgivings about treating proper names as quantifiers in natural-language analysis.³⁴) Lind-





28

The Initial Generalization

ström did not discuss the reasons underlying his extension of Mostowski's system, but as we have just seen, philosophico-linguistic considerations support his approach. In accordance with Lindström's proposal we add to Mostowski's original quantifiers all second-level 2-place relations of firstlevel 1-place predicates satisfying (LQ2). The one-to-one correlation between Mostowski's quantifiers and cardinality functions is preserved under the new extension. (See the appendix.)

The syntax and the semantics of first-order logic with 1- and 2-place generalized quantifiers is a natural extension of the syntax and the semantics of section 2 above. Again, a model for a language of this logic is the same as a model for a language of standard first-order logic: the "enrichment" is expressed by the rules for computing the truth values of formulas in a model (relative to an assignment of elements in the universe to the individual variables of the language).

I will now present a formal description of the extended logic.

First-Order Logic with 1- and 2-Place Generalized Quantifiers

Syntax

Logical symbols In addition to the logical symbols of standard firstorder logic (but with the possible omission of \forall and/or \exists) the language includes 1- and 2-place quantifier symbols: Q_1^1, \ldots, Q_m^1 and Q_1^2, \ldots, Q_n^2 for some positive integers *m* and *n*. (If \forall and/or \exists belong in the language, they fall under the category of 1-place quantifiers, and we add to them the superscript "1.")

Punctuation The usual punctuation symbols for first-order logic plus the symbol",".

Nonlogical symbols The same as in standard first-order logic.

Terms The same as in standard first-order logic.

Formulas The same definition as for standard first-order logic, but the definition of quantified formulas is replaced by the following:

- (1) If Φ is a formula and Q^1 is a 1-place quantifier symbol, then $(Q^1x)\Phi$ is a formula.
- (II) If Φ , Ψ are formulas and Q^2 is a 2-place quantifier symbol, then $(Q^2x)(\Phi, \Psi)$ is a formula.

Semantics

The semantics is the same as that for standard first-order logic, but the definition of satisfaction of quantified formulas in a model \mathfrak{A} with a

universe A relative to an assignment g for the variables of the language is changed to the following:

- (A) If Φ is a formula and Q^1 is a 1-place quantifier symbol, $\mathfrak{A} \models (Q^1 x) \Phi[g]$ iff for some cardinal numbers α and β such that $\alpha + \beta = |\mathcal{A}|$ and $t_{\mathcal{A}}^{Q^1}(\alpha, \beta) = T$, there are exactly α elements $a \in \mathcal{A}$ such that $\mathfrak{A} \models \Phi[g(x/a)]$ and exactly β elements $b \in \mathcal{A}$ such that $\mathfrak{A} \models \sim \Phi[g(x/b)].$
- (B) If Φ , Ψ are formulas and Q² is a 2-place quantifier symbol, $\mathfrak{A} \models (Q^2 x)(\Phi, \Psi)[g]$ iff for some cardinal numbers α , β , γ , δ such that $\alpha + \beta + \gamma + \delta = |A|$ and $t_A^{Q^2}(\alpha, \beta, \gamma, \delta) = T$,
 - there are exactly α elements $a \in A$ such that $\mathfrak{A} \models (\Phi \& \Psi)[g(x/a)]$,
 - there are exactly β elements $b \in A$ such that $\mathfrak{A} \models (\Phi \& \sim \Psi)[g(x/b)]$,
 - there are exactly y elements $c \in A$ such that $\mathfrak{A} \models (\sim \Phi \& \Psi)[g(x/c)]$,
 - there are exactly δ elements $d \in A$ such that $\mathfrak{A} \models (\sim \Phi \& \sim \Psi)[g(x/d)].$

Applications

Within the system defined above we can easily analyze the logical structure of sentences that eluded Mostowski's original devices, like (36) to (39) above. We choose a language that includes, in addition to \exists^1 and M^2 , the 2-place quantifier "60%," defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\alpha + \beta < \aleph_0$ and $\alpha = 60\%(\alpha + \beta)$. We then symbolize (36) to (39) as given below:

- (61) $(M^2x)(x \text{ is an arrow of John's, } x \text{ hits the target})$
- (62) $(60\%^2 x)(x \text{ is a female student in my class, } x \text{ is an A student})$
- (63) $(M^2 x)[x \text{ is a child } \& \sim (\exists^1 y)(x \text{ communicates with } y \text{ during the first} two years of x's life), x is autistic]$
- (64) $(M^2x)[x \text{ is a college, } (M^2y)(y \text{ is a student at } x, \sim (y \text{ is exempt from tuition fees at } x)]$
- By adding to the language the 2-place quantifiers

S², defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\alpha > \gamma$, O², defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\gamma = 0$, N², defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\gamma \neq 0$, F², defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\beta < \gamma$, and R², defined by $t(\alpha, \beta, \gamma, \delta) = T$ iff $\beta > \gamma$,

we can encode the logical structure of sentences (51) to (53) and (57) to (59), which were problematic for Barwise and Cooper:

(65) $(S^2x)(x \text{ is a woman, } x \text{ has been elected to Congress})$

30

- (66) $(O^2x)(x \text{ is a human being, } x \text{ has a brain})$
- (67) a. $\sim (O^2 x)(x \text{ is a man, } x \text{ is allowed in the club})$
 - b. $(N^2x)(x \text{ is a man}, x \text{ is allowed in the club})$
- (68) $(F^2x)(x \text{ is a man}, x \text{ is a woman})$
- (69) $(\mathbb{R}^2 x)(x \text{ is a person } \& x \text{ dies of heart disease, } x \text{ is a person } \& x \text{ dies}$ of cancer)
- (70) $(F^2x)(x \text{ is one of them, } x \text{ is one of us})$

Within the new system we can also represent the logical structure of (50.a) and (50.b) without violating (LQ2):

(71) $(M^2 x)(1 < x < 10, x < 7)$,

(72) $(M^2x)(1 < x < 10, 9 < x < 17)$

We have seen an example of nested 2-place generalized quantifiers in (64). A more concise example is

(73) Most men love most women,

symbolized as

(74) $(M^2x)[x \text{ is a man, } (M^2y)(y \text{ is a woman, } x \text{ loves } y)].$

To formalize (60), we need to include 3-place quantifiers in the system. The reason is that (60) involves comparison between (subsets of) three sets: the set of all boys who took the test, the set of all girls who took the test, and the set of all those who received a perfect score in the test. We have not defined 3-place quantifiers for our formal system, but it is easy to see how this would be done.

A 3-place quantifier is a function that assigns to each universe A a 3-place quantifier on A, Q_A^3 , Q_A^3 is defined by a cardinality function t_A that takes into account all the "atoms" of Boolean combinations (intersections, unions, and complements) of triples, $\langle B, C, D \rangle$, of subsets of A. Since there are eight such atoms: $B \cap C \cap D$, $(B \cap C) - D$, $(C \cap D) - B$, $(D \cap B) - C, B - (C \cup D), C - (B \cup D), D - (B \cup C), \text{ and } A - (B \cup C \cup C)$ D), $t_{A}^{Q^3}$ is a function from 8-tuples, $\langle \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta \rangle$, of cardinal numbers whose sum is |A| to $\{T, F\}$. We need to decide on the order in which α , β , γ , δ , ε , ζ , η , θ represent (sizewise) the atoms generated by B, C, and D in A. I use a Venn diagram to fix a correlation (figure 2.2). We can now formalize (60) as

(75) $(S^3x)(x \text{ is a boy who took the test, } x \text{ is a girl who took the test,}$ x received a perfect score in the test),

where S^3 is defined by a function t such that when |A| is finite,

The Initial Generalization



Figure 2.2

$$\iota(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta) = T \text{ iff } \frac{\alpha + \delta}{\alpha + \beta + \delta + \varepsilon} = \frac{\alpha + \gamma}{\alpha + \beta + \gamma + \zeta}$$

The present theory enables us to explain the syntactic differences between logical and linguistic quantifications, e.g., the linguistic

Every A is B

and the logical

(Every x) $(Ax \rightarrow Bx)$,

by conjecturing that natural language frequently employs quantifiers that. from a purely logical perspective, are redundant. Thus, whereas \forall^2 (defined by: $t(\alpha, \beta, \gamma, \delta) = T$ iff $\beta = 0$ is logically superfluous, it may very well be the quantifier used in natural language in sentences of the form (46). The logical form of (46) is, on this conjecture,

(76) $(Every^2 x)(Ax, Bx)$.

Likewise, we obtain a structure similar to that of

(77) Fewer men than women live to be 80

by employing a 3-place quantifier, F^3 , rather than the 2-place quantifier F^2 . Sentence (77) is then rendered

(78) $(F^3x)(x \text{ is a man}, x \text{ is a woman}, x \text{ lives to be 80})$

instead of the logically simpler

(79) $(F^2x)(x \text{ is a man } \& x \text{ lives to be } 80, x \text{ is a woman } \& x \text{ lives to be } 80).$

F³ is defined, $t(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta) = T$ iff $\delta < \gamma$. Logic with redundant many-place quantifiers can be seen as a bridge between the logical and the grammatical analysis of language.

The linguistic merits of the logical, as opposed to nonlogical, quantifier approach have been noted by linguists working in Logical Form (LF)

research. An anonymous referee for this book indicates that the category of *logical quantifiers* (without the "living on" constraint) is significant not only semantically but also syntactically. Extensive research on LF has shown that "there are systematic syntactic differences between NPs depending upon whether they are logical or non-logical terms." Thus transparency between syntax and semantics favors logical, as opposed to nonlogical, quantifiers. Among linguists whose work exemplifies the logicalquantifier approach are Higginbotham and May (1981) and May (1989, 1990). In addition, May (1991) cites reasons pertaining to language learnability for identifying quantifiers with cardinality operators and categorizing all quantifiers as logical terms. May writes,

In distinguishing the logical elements in the way that we have, we are making cleavage between logical items whose meanings are formally, and presumably exhaustively, determined by UG [Universal Grammar]---the logical terms -- and those whose meanings are underdetermined by UG--the non-logical, or content, words. This makes sense, for to specify the meaning of quantifiers, all that is needed, formally, is pure arithmetic calculation on cardinalities, and there is no reason to think that such mathematical properties are not universal. For other expressions, learning their lexical meanings is determined causally, and will be effected by experience, perception, knowledge, common-sense, etc. But none of these factors is relevant to the meaning of quantifiers. The child has to learn the content of the lexical entries for the non-logical terms, but this is not necessary for the entries for the logical terms, for they are given innately.³⁵

The considerations adduced by May open a way to empirically ground not only the notion of quantifier developed so far but also the philosophical demarcation of logic in general as presented in this book.

A few words about the limitations of Mostowskian quantifiers. Some predicates of natural language are such that a proper representation of their extension is not possible in standard first-order model theory. Quantifier expressions do attach to such predicates, however. Here are two examples:

- (80) Most of the water in the lake has evaporated.
- (81) More arms than we have are needed to win this war.

"Water in the lake" and "arms needed to win this war" do not sort the objects in a universe into those that fall, and those that do not fall under them. Hence the present theory, which does not change the standard structure of first-order models, cannot account for their logical form.

In addition to predicates that defy first-order symbolization, we also find in natural language a use of quantifiers that exceeds the resources of The Initial Generalization

Mostowski's logic. This is the collective, as opposed to the usual, distributive use of quantifiers. Thus the sentence

(82) Five children ate the whole cake

cannot be formalized by

(83) (!5x)(x is a child, x ate the whole cake),

which says that there are exactly five children each of which ate the whole cake. Collective and nonsortal quantifications will not be dwelt on in this book.

6 From Predicative to Relational Quantifiers

The generalized logic with 1- and 2-place quantifiers defined in the last section can easily be extended to a logic with *n*-place quantifiers for any positive integer *n* (Lindström, 1966). With each *n*-place quantifier Q^n we associate a family of cardinality functions t^{Q^n} , which, given a cardinal α , assigns a truth value to each 2^n -tuple of cardinals whose sum is α . Then, given a model \mathfrak{A} with a universe *A* and a sequence of *n* subsets of *A*, $\langle B_1, \ldots, B_n \rangle$, the value of $Q_A^n(B_1, \ldots, B_n)$ depends on whether the atomic Boolean algebra generated by B_1, \ldots, B_n in *A* is such that

$t_A^{\mathbf{Q}^n}(\beta_1,\ldots,\beta_{2^n})=\mathsf{T},$

where $\beta_1, \ldots, \beta_{2^n}$ are the cardinalities of all the atoms of this Boolean algebra ordered in some canonical manner. I will call the *n*-place quantifiers described above *predicative quantifiers* because such quantifiers constitute *n*-place relations among first-level (1-place) *predicates*.

The next step is to consider quantifiers on relations, or *relational quantifiers*. Syntactically, a 1-place relational quantifier is an operator that binds a formula by a sequence of *n* bound variables, $\langle x_1, \ldots, x_n \rangle$, for some finite n > 1. If we change the symbolization of 1- and 2-place predicative quantifiers to Q¹ and Q^{1,1} respectively, we will naturally symbolize 1-place relational quantifiers in *n* variables by Qⁿ. Thus if

$\Phi(x, y)$

is a formula with x and y free, then

$(Q^2x, y)\Phi(x, y)$

is also a formula, generated from $\Phi(x, y)$ by binding the free variables x and y with Q². (The superscript "2" indicates that Q is a 1-place quantifier

over 2-place first-level relations. For 2-place relational quantifiers over n- and m-place relations, in that order, we will use the superscript "n, m.")

Semantically, however, the characterization of logical relational quantifiers is an involved matter. The question is how to interpret the semantic condition (LQ2) with respect to these quantifiers. Recall that (LQ2) stipulates that quantifiers should not distinguish the identity of particular individuals in the universe of a given model. Mostowski construed this condition as requiring that quantifiers be invariant under permutations of the universe. But Mostowski dealt with predicative quantifiers, which semantically are functions on subsets of the universe, and the quantifiers we are dealing with now are relational quantifiers, i.e., functions on subsets of Cartesian products of the universe. If, following Mostowski, we again interpret (LQ2) as invariance under permutations, the question arises, invariance under permutations of what? Should relational quantifiers over a universe A, say 1-place quantifiers over binary relations on A, be invariant under permutations of A? Permutations of $A \times A$? Permutations of $A \times A$ induced in some specified manner by permutations of A? Or should we not interpret (LQ2) in terms of permutations of the universe at all when it comes to relational quantifiers? This question was raised by Higginbotham and May in "Questions, Quantifiers, and Crossing" (1981). From another angle Higginbotham and May ask what is implied by the requirement that quantifiers should not distinguish the *identity* of elements in the universe of discourse.

Yet another question is the relationship between logicality and cardinality. When I earlier discussed Mostowski's generalization, I said that this question could be avoided because on a very natural interpretation of (LQ2), the requirement that logical quantifiers not distinguish the identity of elements in the universe coincides with the requirement that logical quantifiers be definable by cardinality functions. Since (LQ2) is a natural condition on logical quantifiers, the identification of logical-predicative quantifiers with cardinality quantifiers appeared to be justified. However, now that the interpretation of (LQ2) is no longer straightforward, the question of cardinality and logicality has to be tackled directly.

But the question we have to confront first concerns (LQ2) itself. Why should (LQ2) be the semantic condition on logical quantifiers? Neither Mostowski nor Dummett (nor, as I have already indicated, Lindström) have justified their "choice" of invariance under permutations as the characteristic trait of logical quantifiers. So far I too have uncritically accepted their criterion. But in view of the questions we are now facing and in light of the general inquiry we have undertaken in this work, it is

The Initial Generalization

34

now time to rethink the issue of logicality. Without a clear answer to the question of what makes a term *logical*, 1 doubt that we will be able to resolve the uncertainty regarding the correct definition of relational quantifiers. Moreover, a critical analysis of logicality will enable us to evaluate Mostowski's claim—most central to our query—that symbolic logic is *not* exhausted by standard mathematical first-order logic.