Chapter 3
To Be a Logical Term

Since the discovery of generalized quantifiers by A. Mostowski (1957), the question "What is a logical term?" has taken on a significance it did not have before. Are Mostowski's quantifiers "logical" quantifiers? Do they differ in any significant way from the standard existential and universal quantifiers? What logical operators, if any, has he left out? What, in all, are the first- and second-level predicates and relations that can be construed as logical?

One way in which I do not want to ask the question is, "What, *in the nature of things*, makes a property or a relation logical?" On this road lie the controversies regarding necessity and apriority, and these, I believe, should be left aside. Although some understanding of the modalities is essential for our enterprise, only their most general features come into play. A detailed study of complex and intricate modal and epistemic issues would just divert our attention and is of little use here. But if "the nature of things" is not our measure, what is? What should our starting point be? What strategy shall we decide upon?

A promising approach is suggested by L. Tharp in "Which Logic Is the Right Logic?" (1975). Tharp poses the question, What properties should a system of logic have? In particular, is standard first-order logic the "right" logic? To answer questions of this kind, he observes, it is crucial to have a clear idea about "the *role* logic is expected to play."[1] Tharp's point is worth taking, and it provides the clue we are searching for. If we identify a central role of logic and, relative to that role, ask what expressions can function as logical terms, we will have found a perspective that makes our question answerable, and significantly answerable at that.

The most suggestive discussion of the logical enterprise that I know of appears in A. Tarski's early papers on the foundations of semantics. Tarski's papers reveal the forces at work during the inception of modern

logic; at the same time, the principles developed by Tarski in the 1930s are still the principles underlying logic in the early 1990s. My interest in Tarski is, needless to say, not historical. I am interested in the modern conception of logic as it evolved out of Tarski's early work in semantics.

1 The Task of Logic and the Origins of Semantics

In "The Concept of Truth in Formalized Languages" (1933), "On the Concept of Logical Consequence" (1936a), and "The Establishment of Scientific Semantics" (1936b), Tarski describes the semantic project as comprising two tasks:

1. Definition of the *general* concept of truth for formalized languages
2. Definition of the *logical* concepts of truth, consequence, consistency, etc.

The main purpose of (1) is to secure metalogic against semantic paradoxes. Tarski worried lest the uncritical use of semantic concepts prior to his work concealed an inconsistency: a hidden fallacy would undermine the entire venture. He therefore sought precise, materially, as well as formally, correct definitions of "truth" and related notions to serve as a hedge against paradox. This aspect of Tarski's work is well known. In "Model Theory before 1945" R. Vaught (1974) puts Tarski's enterprise in a slightly different light:

> [During the late 1920s] Tarski had become dissatisfied with the notion of truth as it was being used. Since the notion "σ is true in $\mathfrak{A}$" is highly intuitive (and perfectly clear for any definite σ), it had been possible to go even as far as the completeness theorem by treating truth (consciously or unconsciously) essentially as an undefined notion—one with many obvious properties. ... But no one had made an analysis of truth, not even of exactly what is involved in treating it in the way just mentioned. At a time when it was quite well understood that 'all of mathematics' could be done, say, in ZF, with only the primitive notion $\in$, this meant that the theory of models (and hence much of metalogic) was indeed not part of mathematics. It seems clear that this whole state of affairs was bound to cause a lack of sure-footedness in metalogic. ... [Tarski's] major contribution was to show that the notion "σ is true in $\mathfrak{A}$" can simply be *defined* inside of ordinary mathematics, for example, in ZF.[2]

On both accounts the motivation for (1) has to do with the adequacy of the system designed to carry out the logical project, not with the logical project itself. The goal of logic is not the mathematical definition of "true sentence," and (1) is therefore a secondary, albeit crucially important, task of Tarskian logic. (2), on the other hand, does reflect Tarski's vision of the role of logic. In paper after paper throughout the early 1930s Tarski described the logical project as follows:[3] The goal is to develop and study deductive systems. Given a formal system $\mathscr{L}$ with language L and a definition of "meaningful," i.e., "well-formed," sentence for L, a (*closed*) *deductive system* in $\mathscr{L}$ is the set of all logical consequences of some set X of meaningful sentences of L. "Logical consequence" was defined proof-theoretically in terms of logical axioms and rules of inference: if $\mathscr{A}$ and $\mathscr{R}$ are the sets of logical axioms and rules of inference of $\mathscr{L}$, respectively, the set of *logical consequences of X in $\mathscr{L}$* is the smallest set of well-formed sentences of L that includes X and $\mathscr{A}$ and is closed under the rules in $\mathscr{R}$. In contemporary terminology, a deductive system is a *formal theory* within a logical framework $\mathscr{L}$. (Note that the logical framework itself can be viewed as a deductive system, namely by taking X to be the set of logical axioms.) The task of logic, in this picture, is performed in two steps: (a) the construction of a logical framework for formal (formalized) theories; (b) the investigation of the logical properties—consistency, completeness, axiomatizability, etc.—of formal theories relative to the logical framework constructed in step (a). The concept of *logical consequence* (together with that of a well-formed formula) is the key concept of Tarskian logic. Once the definition of "logical consequence" is given, we can easily obtain not only the notion of a deductive system but also those of a logically true sentence; logically equivalent sets of sentences; an axiom system of a set of sentences; and axiomatizability, completeness, and consistency of a set of sentences. The study of the conditions under which various formal theories possess these properties forms the subject matter of metalogic.

Whence semantics? Prior to Tarski's "On the Concept of Logical Consequence" the definitions of the logical concepts were proof-theoretical. The need for semantic definitions of the same concepts arose when Tarski realized that there was a serious gap between the proof-theoretic definitions and the intuitive concepts they were intended to capture: many intuitive consequences of deductive systems could not be detected by the standard system of proof. Thus the sentence "For every natural number n, Pn" seems to follow, in some important sense, from the set of sentences "Pn," where n is a natural number, but there is no way to express this fact by the proof method for standard first-order logic. This situation, Tarski said, shows that proof theory by itself cannot fully accomplish the task of logic. One might contemplate extending the system by adding new rules of inference, but to no avail. Gödel's discovery of the incompleteness of the deductive system of Peano arithmetic showed,

> In every deductive theory (apart from certain theories of a particularly elementary nature), however much we supplement the ordinary rules of inference by new purely structural rules, it is possible to construct sentences which follow, in the usual sense, from the theorems of this theory, but which nevertheless cannot be proved in this theory on the basis of the accepted rules of inference.[4]

Tarski's conclusion was that proof theory provides only a partial account of the logical concepts. A new method is called for that will permit a more comprehensive systematization of the intuitive content of these concepts.

The intuitions underlying our informal notion of logical consequence (and derivative concepts) are anchored, according to Tarski, in certain relationships between linguistic items and objects in (configurations of) the world. The discipline that studies relationships of this kind is called *semantics*:

> We ... understand by semantics the totality of considerations concerning those concepts which, roughly speaking, express certain connexions between the expressions of a language and the objects and states of affairs referred to by these expressions.[5]

The precise formulation of the intuitive content of the logical concepts is therefore a job for semantics. (Although the relation between the set of sentences "Pn" and the universal quantification "$(\forall x)Px$," where x ranges over the natural numbers and "n" stands for a name of a natural number, is not logical consequence, we will be able to characterize it accurately within the framework of Tarskian semantics, e.g., in terms of ω completeness.)

2 The Semantic Definition of "Logical Consequence" and the Emergence of Models

Tarski describes the intuitive content of the concept "logical consequence" as follows:

> Certain considerations of an intuitive nature will form our starting-point. Consider any class K of sentences and a sentence X which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class K consists only of true sentences and the sentence X is false. Moreover, ... we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds. ... The two circumstances just indicated ... seem to be very characteristic and essential for the proper concept of consequence.[6]

We can express the two conditions set by Tarski on a correct definition of "logical consequence" by (C1) and (C2) below:

CONDITION C1 If X is a logical consequence of K, then X is a *necessary* consequence of K in the following intuitive sense: it is impossible that all the sentences of K are true and X is false.

CONDITION C2 Not all necessary consequences fall under the concept of logical consequence; only those in which the consequence relation between a set of sentences K and a sentence X is based on *formal* relationships between the sentences of K and X do.

To provide a formal definition of "logical consequence" based on (C1) and (C2), Tarski introduces the notion of *model*. In current terminology, given a formal system $\mathscr{L}$ with a language L, an *$\mathscr{L}$-model*, or a *model for* $\mathscr{L}$, is a pair, $\mathfrak{A} = \langle A, D \rangle$, where A is a set and D is a function that assigns to the nonlogical primitive constants of L, $t_1, t_2, \ldots$ elements (or constructs of elements) in A: if t_i is an individual constant, $D(t_i)$ is a member of A; if t_i is an n-place first-level predicate, $D(t_i)$ is an n-place relation included in A^n; etc. We will say that the function D assigns to $t_1, t_2, \ldots$ denotations in A. Any pair of a set A and a denotation function D determines a model for $\mathscr{L}$. Given a theory T in a formal system $\mathscr{L}$ with a language L, we say that a model $\mathfrak{A}$ for $\mathscr{L}$ is a *model of* $\mathscr{T}$ iff every sentence of $\mathscr{T}$ is true in $\mathfrak{A}$. (Similarly, $\mathfrak{A}$ is a model of a sentence X of L iff X is true in $\mathfrak{A}$.) The definition of "the sentence X of L is true in a model $\mathfrak{A}$ for $\mathscr{L}$" is given in terms of satisfaction: X is *true in* $\mathfrak{A}$ iff every assignment of elements in A to the variables of L satisfies X in $\mathfrak{A}$. The notion of satisfaction is based on Tarski 1933. I assume that the reader is familiar with this notion.

The formal definition of "logical consequence" in terms of models proposed by Tarski is:

DEFINITION LC The sentence X *follows logically* from the sentences of the class K iff every model of the class K is also a model of the sentence X.[7]

The definition of "logical truth" immediately follows:

DEFINITION LTR The sentence X is *logically true* iff every model is a model of X.

To be more precise, (LC) and (LTR) should be relativized to a logical system $\mathscr{L}$. "Sentence" would then be replaced by "$\mathscr{L}$-sentence" and "model" by "$\mathscr{L}$-model."

(A historical remark is in place here. Some philosophers claim that Tarski's 1936 definition of a model is essentially different from the one currently used because in 1936 Tarski did not require that models vary with respect to their universes. This issue does not really concern us here, since we are interested in the legacy of Tarski, not this or that historical stage in the development of his thought. For the intuitive ideas we go to the early writings, where they are most explicit, while the formal constructions are those that appear in his mature work.

Notwithstanding the above, it seems to me highly unlikely that in 1936 Tarski intended all models to share the same universe. This is because such a notion of model is incompatible with the most important model-theoretic results obtained by logicians, including Tarski himself, before that time. Thus, the Löwenheim-Skolem-Tarski theorem (1915–1928) says that if a first-order theory has a model with an infinite universe A, it has a model with a universe of cardinality α for every infinite α. Obviously, this theorem does not hold if one universe is common to all models. Similarly, Gödel's 1930 completeness theorem fails: if all models share the same universe, then for every positive integer n, one of the two first-order statements "There are more than n things" and "There are at most n things" is true in all models, and hence, according to (LTR), it is logically true. But no such statement is provable from the logical axioms of standard first-order logic.[8] Be that as it may, the Tarskian concept of model discussed here does include the requirement that any nonempty set is the universe of some model for the given language.)

Does (LC) satisfy the intuitive requirements on a correct definition of "logical consequence" given by (C1) and (C2) above? According to Tarski it does:

> It seems to me that everyone who understands the content of the above definition must admit that it agrees quite well with common usage. . . . It can be proved, on the basis of this definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences.[9]

In what way does (LC) satisfy (C1)? Tarski mentions the existence of a proof but does not provide a reference. There is a very simple argument that, I believe, is in the spirit of Tarski:[10]

Proof Assume X is a logical consequence of K, i.e., X is true in all models in which all the members of K are true. Suppose that X is not a necessary consequence of K. Then it is possible that all the members of K are true and X is false. But in that case there is a model in which all the members of K come out true and X comes out false. Contradiction.

The argument is simple. However, it is based on a crucial assumption:

ASSUMPTION AS If K is a set of sentences and X is a sentence (of a formal language L of $\mathscr{L}$) such that it is intuitively possible that all the members of K are true while X is false, then there is a model (for $\mathscr{L}$) in which all the members of K come out true and X comes out false.

Assumption (AS) is equivalent to the requirement that, given a logic $\mathscr{L}$ with a formal language L, every possible state of affairs relative to the expressive power of L be represented by some model for $\mathscr{L}$. (Note that (AS) does not entail that every state of affairs represented by a model for $\mathscr{L}$ is possible. This accords with Tarski's view that the notion of *logical possibility* is weaker than, and hence different from, the general notion of possibility [see (C2)].) Is (AS) fulfilled by Tarski's model-theoretic semantics?

We can show that (AS) holds at least for standard first-order logic. Let $\mathscr{L}$ be a standard first-order system, L the language of $\mathscr{L}$, K a set of sentences of L, and X a sentence of L. Suppose it is intuitively possible that all the members of K are true and X is false. Then, if we presume that the rules of inference of standard first-order logic are necessarily truth-preserving, $K \cup \{\sim X\}$ is intuitively consistent in the proof-theoretic sense: for no first-order sentence Y are both Y and $\sim Y$ provable from $K \cup \{\sim X\}$. It follows from the completeness theorem for first-order logic that there is a model for $\mathscr{L}$ in which all the sentences of K are true and X is false.

As for (C2), Tarski characterizes the formality requirement as follows:

> Since we are concerned here with the concept of logical, i.e., *formal* consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences of the class K refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects.[11]

The condition of formality, (C2), has several aspects. First, logical consequences, according to Tarski, are based on the logical form of the sentences involved. The logical form of sentences is in turn determined by their logical terms (see Tarski's notion of a well-formed formula in "The Concept of Truth in Formalized Languages"). Therefore, logical consequences are based on the logical terms of the language. Second, logical consequences are not empirical. This means that logical terms, which determine logical consequences, are not empirical either. Finally, logical consequences "cannot be affected by replacing the designations of the objects ... by other objects." In "The Concept of Logical Consequence" Tarski first attempted a substitutional interpretation of the last requirement. This led to a substitutional definition of "logical consequence." According to this definition, consequences preserved under all (uniform, type preserving) substitutions of the nonlogical terms of the language are logical. However, Tarski soon realized that the substitutional definition did not capture the notion of logical consequence in all its generality.[12] The substitutional test depends on the expressive power of the language in question. In particular, languages with a meager vocabulary of singular terms let intuitively nonlogical consequences pass for genuinely logical ones. Tarski's reaction to the shortcomings of the substitutional test was to drop the idea of substitutivity altogether. Instead, Tarski turned to *semantics*, a new discipline devoted to studying the relation between language and the world, whose basic notions are "satisfaction" and "model." On the basis of these concepts Tarski proposed the model-theoretic definition of logical consequence, (LC). Although Tarski did not explain what "indifference of the consequence relation to replacement of objects" meant semantically, I think we can offer the following analysis inspired by Mostowski. There are terms that take the identity of objects into account and terms that do not. Terms underlying logical consequences must be of the second kind. That is to say, logical terms should not distinguish the identity of objects in the universe of any model. (By "identity of an object" I here mean the features that make an object what it is, the properties that single it out.)

Now clearly Tarskian consequences of standard first-order logic satisfy the formality condition. First, only entirely trivial consequences (X follows logically from K just in case $X \in K$) obtain without logical terms. Therefore, logical consequences are due to logical terms of the language. Second, the truth-functional connectives, identity, and the universal and existential quantifiers are nonempirical functions that do not distinguish the objects in any given model. The substitution test, which is still necessary (though not sufficient), is also passed by standard logic.

We see that (C2), the condition of formality, sets a limit on (C1), the condition of necessity: necessity does not suffice for logicality. While all logical consequences are necessary, only necessary consequences that are also formal count as genuinely logical. An example of a necessary consequence that fails to satisfy the condition of formality is,

(1) *b* is red all over; therefore *b* is not blue all over.

This consequence is not logical according to Tarski's criterion, because it hangs on particular features of color properties that depend on the identity of objects in the universe of discourse. (Try to replace "blue" with "smooth," a replacement that has no bearing on the formal relations between premise and conclusion, and see what happens.) Later we will also see that (C1) sets a restriction on the application of (C2).

I think conditions (C1) and (C2) on the key concept of logical consequence delineate the scope as well as the limit of Tarski's enterprise: the development of a conceptual system in which the concept of logical consequence ranges over *all* formally necessary consequences and nothing else. Since our intuitions leave some consequences undetermined with respect to formal necessity, the boundary of the enterprise is somewhat vague. But the extent of vagueness is limited. Formal necessity is a relatively unproblematic notion, and the persistent controversies involving the modalities are not centered around the formal.

We have seen that at least in one application, namely, in standard first-order logic, Tarski's definition of logical consequence stands the test of (C1) and (C2): all the standard consequences that fall under Tarski's definition are indeed formal and necessary. We now ask, Does standard first-order logic yield *all* the formally necessary consequences with a first-level (extensional) vocabulary? Could not the standard system be extended so that Tarski's definition encompasses new consequences satisfying the intuitive conditions but undetected within the standard system? Tarski himself all but asked the same question. He ended "On the Concept of Logical Consequence" with the following note:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage.[13]

The question, "What is the full scope of logic?" I will ask in the form: What is the widest notion of a *logical term* for which the Tarskian definition of "logical consequence" gives results compatible with (C1) and (C2)?

3 Logical and Extralogical Terms: An Unfounded Distinction?

What is the widest definition of "logical term" compatible with Tarski's theory? In 1936 Tarski did not know how to handle the problem of new logical terms. Tarski's interest was not in extending the scope of "logical

consequence" but in defining this concept successfully for standard logic. From this point of view, the relativization of "logical consequence" to collections of logical terms was disquieting. While Tarski's definition produced the right results when applied to standard first-order logic, there was no guarantee that it would continue to do so in the context of wider "logics." A standard for logical terms could solve the problem, but Tarski had no assurance that such a standard was to be found. The view that Tarski's notion of logical consequence is inevitably tied up with arbitrary choices of logical terms was advanced by J. Etchemendy (1983, 1990). Etchemendy was quick to point out that this arbitrary relativity undermines Tarski's theory. I will not discuss Etchemendy's interpretation of Tarski here, but I would like to examine the issue in the context of my own analysis. Is the distinction between logical and extralogical terms founded? If it is, what is it founded on? Which term falls under which category?

Tarski did not see where to draw the line. In 1936 he went as far as saying that "in the extreme case we could regard all terms of the language as logical. The concept of *formal* consequence would then coincide with that of *material* consequence."[14] Unlike "logical consequence," the concept of material consequence is defined without reference to models:

DEFINITION MC The sentence X is a *material consequence* of the sentences of the class K iff at least one sentence of K is false or X is true.[15]

Tarski's statement first seemed to me clear and obvious. However, on second thought I found it somewhat puzzling. How could *all* material consequences of a hypothetical first-order logic $\mathscr{L}$ become logical consequences? Suppose $\mathscr{L}$ is a logic in which "all terms are regarded as logical." Then evidently the standard logical constants are also regarded as logical in $\mathscr{L}$. Consider the $\mathscr{L}$-sentence:

(2) There is exactly one thing,

or, formally,

(3) $(\exists x)(\forall y) x = y$.

This sentence is false in the real world, hence

(4) There are exactly two things

follows materially from it (in $\mathscr{L}$). But Tarski's semantics demands that for each cardinality α, there be a model for $\mathscr{L}$ with a universe of cardinality α. (This much comes from his requirement that any arbitrary set of objects constitute the universe of some model for $\mathscr{L}$). Thus in particular $\mathscr{L}$ has a model with exactly one individual. It is therefore not true that in every model in which (2) is true, (4) is true too. Hence, according to Tarski's definition, (4) is not a logical consequence of (2).

So Tarski conceded too much: no addition of new logical terms would trivialize his definition altogether. Tarski underestimated the viability of his system. His model-theoretic semantics has a built-in barrier that prevents a total collapse of logical into material consequence. To turn all material consequences of a given formal system $\mathscr{L}$ into logical consequences requires limiting the totality of sets in which $\mathscr{L}$ is to be interpreted. But the requirement that no such limit be set is intrinsic to Tarski's notion of a model.

It appears, then, that what Tarski had to worry about was not total but partial collapse of logical into material consequence. However, it is still not clear what "regarding all the terms of the language as logical" meant. Surely Tarski did not intend to say that if all the constant terms of a logic $\mathscr{L}$ are logical, the distinction between formal and material consequence for $\mathscr{L}$ collapses. The language of pure identity is a conspicuous counterexample. All the constant terms of that language are logical, yet the definition of "logical consequence" yields a set of consequences different (in the right way) from the set of material consequences.

We should also remember that Tarski's definition of "logical consequence" and the definition of "satisfaction" on which it is based are applicable only to formalized languages whose vocabulary is essentially restricted. Therefore, Tarski could not have said that if we regard all terms of natural language as logical, the definition of "logical consequence" will coincide with that of "material consequence". A circumstance concerning natural language in its totality could not have any effect on the Tarskian concept of logical consequence.

Even with respect to single constants it is not altogether clear what treating them as logical might mean. Take, for instance, the term "red." How do you construe "red" as a logical constant? To answer this question we have to find out what makes a term logical (extralogical) in Tarski's system. Only then will we be able to determine whether any term whatsoever can be regarded as logical in Tarski's logic.

4 The Roles of Logical and Extralogical Terms

What makes a term logical or extralogical in Tarski's system? Considering the question from the "functional" point of view I have opted for, I ask: How does the dual system of a formal language and its model-theoretic semantics accomplish the task of logic? In particular, what is the role

of logical and extralogical constants in determining logical truths and consequences?

Extralogical constants

Consider the statement

(5) Some horses are white,

formalized in standard first-order logic by

(6) $(\exists x)(Hx \& Wx)$.

How does Tarski succeed in giving this statement truth conditions that, in accordance with our clear pretheoretical intuitions, render it logically indeterminate (i.e., neither logically true nor logically false)? The crucial point is that the common noun "horse" and the adjective "white" are interpreted within models in such a way that their intersection is empty in some models and not empty in others. Similarly, for any natural number n, the sentence

(7) There are n white horses

is logically indeterminate because in some but not all models "horse" and "white" are so interpreted as to make their intersection of cardinality n. Were "finitely many" expressible in the logic, a similar configuration would make

(8) Finitely many horses are white

logically indeterminate as well.

In short, what is special to extralogical terms like "horse" and "white" in Tarskian logic is their *strong semantic variability*. Extralogical terms have no independent meaning: they are interpreted only *within* models. Their meaning in a given model is nothing more than the value that the denotation function D assigns to them in that model. We cannot speak about *the* meaning of an extralogical term: being extralogical implies that nothing is ruled out with respect to such a term. Every denotation of the extralogical terms that accords with their syntactic category appears in some model. Hence the totality of interpretations of any given extralogical term in the class of all models for the formal system is exactly the same as that of any other extralogical term of the same syntactic category. Since every set of objects is the universe of some model, any possible state of affairs — any possible configuration of individuals, properties, relations, and functions — via-à-vis the extralogical terms of a given formalized language (possible, that is, with respect to their meaning prior to formalization) is represented by some model.

Formally, we can define Tarskian extralogical terms as follows:

DEFINITION ET $\{e_1, e_2, \ldots\}$ is the set of primitive *extralogical terms* of a Tarskian logic $\mathscr{L}$ iff for every set A and every function D that assigns to $e_1, e_2, \ldots$ denotations in A (in accordance with their syntactic categories), there is a model $\mathfrak{A}$ for $\mathscr{L}$ such that $\mathfrak{A} = \langle A, D\rangle$.

It follows from (ET) that primitive extralogical terms are semantically unrelated to one another. As a result, complex extralogical terms, produced by intersections, unions, etc. of primitive extralogical terms (e.g., "horse and white") are strongly variable as well.

Note that it is essential to take into account the strong variability of extralogical terms in order to understand the meaning of various claims of logicality. Consider, for instance, the statement

(9) $(\exists x)x =$ Jean-Paul Sartre,

which is logically true in a Tarskian logic with "Jean-Paul Sartre" as an extralogical individual constant. Does the claim that (9) is logically true mean that the existence (unspecified with respect to time) of the deceased French philosopher is a matter of logic? Obviously not. The logical truth of (9) reflects the principle that if a term is used in a language to name objects, then in every model for the language some object is named by that term. But since "Jean-Paul Sartre" is a strongly variable term, what (9) says is "There is *a* Jean-Paul Sartre," not "*The* (French philosopher) Jean-Paul Sartre exists."

Logical constants

It has been said that to be a logical constant in a Tarskian logic is to have *the same* interpretation in all models. Thus for "red" to be a logical constant in logic $\mathscr{L}$, it has to have a constant interpretation in all the models for $\mathscr{L}$. I think this characterization is faulty because it is vague. How do you interpret "red" *in the same way* in all models? "In the same way" in what sense? Do you require that in every model there be the same number of objects falling under "red"? But for every number larger than 1 there is a model that cannot satisfy this requirement simply because it does not have enough elements. So at least in one way, cardinalitywise, the interpretation of "red" must vary from model to model.

The same thing holds for the standard logical constants of Tarskian logic. Take the universal quantifier. In every model for a first-order logic the universal quantifier is interpreted as a singleton set (i.e., the set of the universe).[16] But in a model with 10 elements it is a set of a set with 10 elements, whereas in a model with 9 elements it is a set of a set with 9 elements. Are these interpretations the same?[17]

I think that what distinguishes logical constants in Tarski's semantics is not the fact that their interpretation does not vary from model to model (it does!) but the fact that they are interpreted *outside* the system of models.[18] The meaning of a logical constant is not given by the definitions of particular models but is part of the same metatheoretical machinery used to define the entire network of models. The meaning of logical constants is given by *rules external to the system*, and it is due to the existence of such rules that Tarski could give his recursive definition of truth (satisfaction) for well-formed formulas of any given language of the logic. Syntactically, the logical constants are "fixed parameters" in the inductive definition of the set of well-formed formulas; semantically, the rules for the logical constants are the functions on which the definition of satisfaction by recursion (on the inductive structure of the set of well-formed formulas) is based.

How would different choices of logical terms affect the extension of "logical consequence"? Well, if we contract the standard set of logical terms, some intuitively formal and necessary consequences (i.e., certain logical consequences of standard first-order logic) will turn nonlogical. If, on the other hand, we take any term whatsoever as logical, we will end up with new "logical" consequences that are intuitively not formally necessary. The first case does not require much elaboration: if "and" were interpreted as "or," X would not be a logical consequence of "X and Y." As for the second case, let us take an extreme example. Consider the natural-language terms "Jean-Paul Sartre" and "accepted the Nobel Prize in literature," and suppose we use them as logical terms in a Tarskian logic by keeping their usual denotation "fixed." That is, the semantic counterpart of "Jean-Paul Sartre" will be the existentialist French philosopher Jean-Paul Sartre, and the semantic counterpart of "accepted the Nobel Prize in literature" will be the set of all actual persons up to the present who (were awarded and) accepted the Nobel Prize in literature. Then

(10) Jean-Paul Sartre accepted the Nobel Prize in literature

will come out false, according to Tarski's rules of truth (satisfaction), no matter what model we are considering. This is because, when determining the truth of (10) in any given model $\mathfrak{A}$ for the logic, we do not have to look in $\mathfrak{A}$ at all. Instead, we examine two fixed entities outside the apparatus of models and determine whether the one is a member of the other. This renders (10) logically false, and according to Tarski's definition, any sentence of the language we are considering follows logically from it, in contradiction with the pretheoretical conditions (C1) and (C2).

The above example violates two principles of Tarskian semantics: (1) "Jean-Paul Sartre" and "accepted the Nobel Prize in literature" do not satisfy the requirement of formality. (2) The truth conditions for (10) bypass the very device that serves in Tarskian semantics to distinguish material from logical consequence, namely the apparatus of models. No wonder the definition of "logical consequence" fails!

It is easy to see that each violation by itself suffices to undermine Tarski's definition. In the case of (1), "Jean-Paul Sartre" and "accepted the Nobel Prize in literature" are empirical terms that do distinguish between different objects in the universe of discourse. As for (2), suppose we define logical terms in accordance with (C2) but without reference to the totality of models. Say we interpret the universal quantifier for a single universe, that of the natural numbers. In that case "for every" becomes "for every natural number," and the statement

(11) Every object is different from at least three other objects

turns out logically true, in violation of the intuition embedded in (C1). By requiring that "every" be defined over all models, we circumvent the undesirable result.

We can now see how Tarski's method allows us to identify a sentence like

(12) Everything is identical with itself

as the logical truth that it intuitively is. The crucial point is that the intuitive meanings of "is identical with" and "everything" are captured by rules definable over all models. These rules single out pairs and sets of objects that share certain formal features which do not vary from one possible state of affairs to another. Thus in all models (representations of possible states of affairs), the set of self-identical objects is universal (i.e., coincides with the universe), and in each model the universal set is "everything" for that particular model.

5 The Distinction between Logical and Extralogical Terms: A Foundation

The discussion of logical and extralogical terms enables us to answer the questions posed in section 3. We understand what it means to regard *all*

terms of the language as logical. Within the scheme of Tarski's logic it means to allow any rule whatsoever to be the semantic definition of a logical constant. In particular, the intuitive interpretation of any term becomes its semantic rule qua a logical term. Our investigation clearly demonstrated that not every interpretation of logical terms is compatible with Tarski's vision of the task of logic.

We can now turn to the main question of section 3. Is the distinction between logical and extralogical terms founded? Of course it is! The distinction between logical and extralogical terms is founded on our pretheoretical intuition that logical consequences are distinguished from material consequences in being necessary and formal. To reject this intuition is to drop the foundation of Tarski's logic. To accept it is to provide a ground for the division of terms into logical and extralogical.

But what is the boundary between logical and extralogical terms? Should we simply say that a constant is logical if adding it to the standard system would not conflict with (C1) and (C2)? This criterion is correct but not very informative. It appears that consequences like

(13) Exactly one French philosopher refused the Nobel Prize in literature; therefore, finitely many French philosophers did

are formal and necessary in Tarski's sense. Therefore "finitely many" is a reasonable candidate for logical constanthood. But can we be sure that "finitely many" will never lead to a conflict with (C1) and (C2)? And will our intuitions guide us in each particular case? By themselves, (C1) and (C2) do not provide a usable criterion. Let us see if their analysis in the context of Tarski's system will not lead us to the desired criterion.

The view that logic is an instrument for identifying formal and necessary consequences leads to two initial requirements (based on (C1) and (C2)): (1) that every possible state of affairs vis-à-vis a given language be represented by some model for the language, and (2) that logical terms represent formal features of possible states of affairs, i.e., formal properties of (relations among) constituents of states of affairs. To satisfy these requirements the Tarskian logician constructs a dual system, each part of which is itself a complex, syntactic-semantic structure. One constituent includes the extralogical vocabulary (syntax) and the apparatus of models (semantics). I will call it the *base* of the logic. (Note that only extralogical terms, not logical terms, play a role in constructing models.) In a first-order logic the base is strictly first-level: syntactically, the extralogical vocabulary includes only singular terms and terms whose arguments are singular; semantically, in any given model the extralogical terms are assigned only individuals or sets, relations, and functions of individuals.

The second part consists of the logical terms and their semantic definitions. Its task is to introduce formal structure into the system. Syntactically, logical terms are formula-building operators; semantically, they are assigned pre-fixed functions on models that express formal properties of, relations among, and functions of "elements of models" (objects in the universe and constructs of these). Since logical terms are meant to represent formal properties of elements of models corresponding to the extralogical vocabulary, their level is generally higher than that of nonlogical terms. Thus in standard first-order logic, identity is the only first-level logical term. The universal and existential quantifiers are second level, semantically as well as syntactically, and the logical connectives too are of higher level. As for singular terms, these can never be construed as logical. This is because singular terms represent atomic components of models, and atomic components, being atomic, have no structure (formal or informal). I will say that the system of logical terms constitutes a *superstructure* for the logic.

The whole system is brought together by superimposing the logical apparatus on the nonlogical base. Syntactically, this is done by rules for forming well-formed formulas by means of the logical operators, and semantically, by rules for determining truth (satisfaction) in a model based on the formal denotations of the logical vocabulary. (Note that since the systems we are considering are extensional, "interpretation" has the same import as "denotation.")

Now, to satisfy the conditions (C1) and (C2), it is essential that no logical term represent a property or a relation that is intuitively variable from one state of affairs to another. Furthermore, it is important that logical terms be formal entities. Finally, the denotations of logical terms need to be defined over models, all models, so that every possible state of affairs is taken into account in determining logical truths and consequences.

It appears that if we can specify a series of conditions that are exclusively and exhaustively satisfied by terms fulfilling the requirements above, we will have succeeded in defining "logical term" in accordance with Tarski's basic principles. In particular, the Tarskian definition of "logical consequence" (and the other metalogical concepts) will give correct results, all the correct results, in agreement with (C1) and (C2).

6 A Criterion for Logical Terms

My central idea is this. Logical terms are formal in a sense that was specified in section 2. There we already interpreted the requirement of formality in the spirit of Mostowski as "not distinguishing the identity of objects in a given universe." Why don't we take another step in the same direction and follow Mostowski's construal of "not distinguishing the identity of objects" as invariance under permutations (see chapter 2). Generalizing Mostowski, we arrive at the notion of a logical term as formal in the following sense: being formal is, semantically, being invariant under all nonstructural variations of models. That is to say, being formal is being invariant under isomorphic structures. In short, logical terms are *formal* in the sense of being essentially *mathematical*. Since, intuitively, the mathematical parameters of reality do not vary from one possible state of affairs to another, the claim that logical consequences are intuitively necessary is in principle satisfied by logics that allow mathematical terms as logical terms. My thesis, therefore, is this: all and only formal terms, terms invariant under isomorphic structures, can serve as logical terms in a logic based on Tarski's ideas. I must, however, add the proviso that new terms be incorporated in the logical system "in the right way."

I will now proceed to set down in detail the criterion for logical terms. But first let me make a few preliminary remarks. My analysis of Tarski's syntactic-semantic system did not depend on the particulars of the metatheoretic language in which the syntax and the semantics are embedded. In standard mathematical logic the metalanguage consists of a fragment of natural language augmented by first-order set theory or higher-order logic. In particular, models are set-theoretic constructs, and the definition of "satisfaction in a model" is accordingly set-theoretical. This feature of contemporary metalogic is, however, not inherent in the nature of the logical enterprise, and one could contemplate a background language different from the one currently used. Without committing myself to any particular metatheoretical mathematics, I will nevertheless use the terminology of standard first-order set theory in the formal entries of the criterion for logical terms, as this will contribute to precision and clarity.

For transparency I will not include sentential connectives in the criterion. While it is technically easy to construe the connectives as quantifiers (see Lindström 1966), the syntactic-semantic apparatus of Tarskian logic is superfluous for analyzing their scope. The standard framework of sentential logic is perfectly adequate, and relative to this framework, the problem of identifying all the logical connectives that there are has already been solved. The solution clearly satisfies Tarski's requirements: the standard logic of sentential connectives has a base that consists syntactically of extralogical sentential letters and semantically of a list of all possible assignments of truth values to these letters. Any possible state of affairs vis-à-vis the sentential language is represented by some assignment. The logical superstructure includes the truth-functional connectives and their semantic definitions. The connectives are both syntactically and semantically of a higher level than the sentential letters. Their semantic definitions are pre-fixed: logical connectives are semantically identified with truth-functional operators, and the latter are defined by formal (Boolean) functions whose values and arguments, i.e., truth values and sequences of truth values, represent possible states of affairs. This ensures that truths and consequences that hold in all "models" are formally necessary in Tarski's sense.

As for modal operators, they too are outside the scope of this investigation, though for different reasons. First, my criterion for logical terms is based on analysis of the Tarskian framework, which is insufficient for modals. Second, we cannot take it for granted that the task of modal logic is the same as that of symbolic logic proper. To determine the scope of modal logic and characterize its operators, we would have to set upon an independent inquiry into its underlying goals and principles.

Conditions on logical constants in first-order logics

The criterion for logical terms based on the Tarskian conception of formal first-order logic will be formulated in a series of individually necessary and collectively sufficient conditions. These conditions will specify what simple and/or complex terms from an initial pool of constants can serve as logical constants in a first-order logic. In stating these conditions, I place a higher value on clarity of ideas than on economy. As a result the conditions are not mutually independent.

A. A logical constant C is syntactically an n-place predicate or functor (functional expression) of level 1 or 2, n being a positive integer.

B. A logical constant C is defined by a single extensional function and is identified with its extension.

C. A logical constant C is defined over models. In each model $\mathfrak{A}$ over which it is defined, C is assigned a construct of elements of $\mathfrak{A}$ corresponding to its syntactic category. Specifically, I require that C be defined by a function f_c such that given a model $\mathfrak{A}$ (with universe A) in its domain:

   a. If C is a first-level n-place predicate, then $f_c(\mathfrak{A})$ is a subset of A^n.

   b. If C is a first-level n-place functor, then $f_c(\mathfrak{A})$ is a function from A^n into A.

   c. If C is a second-level n-place predicate, then $f_c(\mathfrak{A})$ is a subset of $B_1 \times \cdots \times B_n$, where for $1 \le i \le n$,

$$B_i = \begin{cases} A & \text{if } i(C) \text{ is an individual} \\ P(A^m) & \text{if } i(C) \text{ is an } m\text{-place predicate} \end{cases}$$

   ($i(C)$ being the ith argument of C).

   d. If C is a second-level n-place functor, then $f_c(\mathfrak{A})$ is a function from $B_1 \times \cdots \times B_n$ into B_{n+1}, where for $1 \le i \le n+1$, B_i is defined as in (c).

D. A logical constant C is defined over *all* models (for the logic).

E. A logical constant C is defined by a function f_c which is invariant under isomorphic structures. That is, the following conditions hold:

   a. If C is a first-level n-place predicate, $\mathfrak{A}$ and $\mathfrak{A}'$ are models with universes A and A' respectively, $\langle b_1, \ldots, b_n \rangle \in A^n$, $\langle b'_1, \ldots, b'_n \rangle \in A'^n$, and the structures $\langle A, \langle b_1, \ldots, b_n \rangle\rangle$ and $\langle A', \langle b'_1, \ldots, b'_n \rangle\rangle$ are isomorphic, then $\langle b_1, \ldots, b_n \rangle \in f_c(\mathfrak{A})$ iff $\langle b'_1, \ldots, b'_n \rangle \in f_c(\mathfrak{A}')$.

   b. If C is a second-level n-place predicate, $\mathfrak{A}$ and $\mathfrak{A}'$ are models with universes A and A' respectively, $\langle D_1, \ldots, D_n \rangle \in B_1 \times \cdots \times B_n$, $\langle D'_1, \ldots, D'_n \rangle \in B'_1 \times \cdots \times B'_n$ (where for $1 \le i \le n$, B_i and B'_i are as in (C.c)), and the structures $\langle A, \langle D_1, \ldots, D_n \rangle\rangle$, $\langle A', \langle D'_1, \ldots, D'_n \rangle\rangle$ are isomorphic, then $\langle D_1, \ldots, D_n \rangle \in f_c(\mathfrak{A})$ iff $\langle D'_1, \ldots, D'_n \rangle \in f_c(\mathfrak{A}')$.

   c. Analogously for functors.

Some explanations are in order. Condition (A) reflects the perception of logical terms as structural components of the language. In particular, it rules out individual constants as logical terms. Note, however, that although an individual by itself cannot be represented by a logical term (since it lacks "inner" structure), it can combine with functions, sets, or relations to form a structure representable by a logical term. Thus, below I define a logical constant that represents the structure of the natural numbers with their ordering relation and zero (taken as an individual). The upper limit on the level of logical terms is 2, since the logic we are considering is a logic for first-level languages, and a first-level language can only provide its logical terms with arguments of level 0 or 1.

Condition (B) ensures that logical terms are *rigid*. Each logical term has a *pre-fixed* meaning in the metalanguage. This meaning is unchangeable and is completely exhausted by its semantic definition. That is to say, from the point of view of Tarskian logic, there are no "possible worlds" of logical terms. Thus, qua logical terms, the expressions "the number of planets" and "9" are indistinguishable. If you want to express the intuition that the number of planets changes from one possible "world" to another, you have to construe it as an extralogical term. If, on the other hand, you choose to use it as a logical term (or in the definition of a logical term), only its extension counts, and this is the same as the extension of "9."

Condition (C) provides the tie between logical terms and the apparatus of models. By requiring that logical terms be defined by fixed functions from models to structures within models, it allows logical terms to represent "fixed" parameters of changeable states of affairs. By requiring that logical terms be defined for each model by elements of this model, it ensures that the apparatus of models is not bypassed when logical truths and consequences are determined. Condition (C) also takes care of the correspondence in categories between the syntax and the semantics.

The point of (D) is to ensure that *all* possible states of affairs are taken into account in determining logical truths and consequences. Thus truth-in-all-models is *necessary* truth and consequence-in-all-models is *necessary* consequence. Conditions (B) to (D) together express the requirement that logical terms are semantically superimposed on the apparatus of models.

With (E) I provide a general characterization of formality: to be *formal* is *not to distinguish between (to be invariant under) isomorphic structures*. This criterion is almost universally accepted as capturing the intuitive (semantic) idea of formality. I will trace the origins of condition (E) and discuss its significance separately in section 7 below. It follows from (E) that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are models with the same universe A, then for every logical term C, $f_C(\mathfrak{A}_1) = f_C(\mathfrak{A}_2)$. Therefore, we can treat logical terms as functions on universes (sets) rather than models, i.e., use $f_C(A)$ instead of $f_C(\mathfrak{A})$. I will do so in chapter 4, using C_A and $C_{\mathfrak{A}}$ as abbreviations.[19]

I can now give a semantic definition of (Tarskian) logical terms:

DEFINITION LT C is a (Tarskian) *logical term* iff C is a truth-functional connective or C satisfies conditions (A) to (E) above on logical constants.

I will call logical terms of the types (C.a) and (C.b) above *logical predicates* and *logical functors* respectively. Logical terms of type (C.c) I will call *logical quantifiers*, and logical terms of type (C.d) *logical quantifier functors*.

What kind of expressions satisfy (LT)? Clearly, all the logical constants of standard first-order logic do. Identity and the standard quantifiers are defined by total functions f_I, $f_\forall$, and $f_\exists$ on models such that, given a model $\mathfrak{A}$ with universe A,

(14) $f_I(\mathfrak{A}) = \{\langle a, b\rangle : a, b \in A \ \&\ a = b\}$,

(15) $f_\forall(\mathfrak{A}) = \{B : B = A\}$,

(16) $f_\exists(\mathfrak{A}) = \{B : B \subseteq A \ \&\ B \neq \varnothing\}$.

The definitions of the truth-functional connectives remain unchanged. Among the nonstandard terms satisfying (LT) are all Mostowskian quantifiers. As explained in chapter 2, these are n-place *predicative* quantifiers, i.e., quantifiers over n-tuples of predicates (where n is a positive integer, and a 1-tuple of predicates is a predicate). Among these are the following, redefined in the style of conditions (A) to (E).

(17) The 1-place "cardinal" quantifiers, defined, for any cardinal α by

$f_\alpha(\mathfrak{A}) = \{B : B \subseteq A \ \&\ |B| = \alpha\}$

(18) The 1-place quantifiers "finitely many" and "uncountably many," defined by

$f_{\text{finite}}(\mathfrak{A}) = \{B : B \subseteq A \ \&\ |B| < \aleph_0\}$

$f_{\text{uncountably many}}(\mathfrak{A}) = \{B : B \subseteq A \ \&\ |B| > \aleph_0\}$

(19) The 1-place quantifier "as many as not," defined by

$f_{\text{as many as not}}(\mathfrak{A}) = \{B : B \subseteq A \ \&\ |B| \geq |A - B|\}$

(20) The 1-place quantifier "most," defined by

$f_{M^1}(\mathfrak{A}) = \{B : B \subseteq A \ \&\ |B| > |A - B|\}$

(21) The 2-place quantifier "most," defined by

$f_{M^{1,1}}(\mathfrak{A}) = \{\langle B, C\rangle : B, C \subseteq A \ \&\ |B \cap C| > |B - C|\}$

We also have *relational* quantifiers satisfying (LT). One of these is,

(22) The "well-ordering" quantifier (a 1-place quantifier over 2-place relations), defined by $f_{WO}(\mathfrak{A}) = \{R : R \subseteq A^2 \ \&\ R$ is a strict linear ordering such that every nonempty subset of Fld(R) has a minimal element in $R\}$.

I will call the logical terms below "relational quantifiers" as well:

(23) The second-level set-membership relation (a 2-place quantifier over pairs of a singular term and a predicate), defined by

$f_{\text{membership}}(\mathfrak{A}) = \{\langle a, B\rangle : a \in A \ \&\ B \subseteq A \ \&\ a \in B\}$

(24) The quantifier "ordering of the natural numbers with 0" (a 2-place quantifier over pairs of a 2-place relation and a singular term), defined by $f_{>_N,0}(\mathfrak{A}) = \{\langle R, a\rangle : R \subseteq A^2 \;\&\; a \in A \;\&$ and $\langle A, R, a\rangle$ is a structure of the natural numbers with their ordering relation and zero$\}$

Among *functors* and *quantifier functors* we have the following:

(25) The *n*-place "first" functors (over *n*-tuples of singular terms), defined, for any *n*, by $f_{\text{first}}(\mathfrak{A})$ = the function $g : A^n \to A$ such that for any *n*-tuple $\langle a_1, \ldots, a_n\rangle \in A^n$, $g(a_1, \ldots, a_n) = a_1$

(26) The 1-place "complement" quantifier functor (over 1-place predicates), defined by $f_{\text{complement}}(\mathfrak{A})$ = the function $g : \boldsymbol{P}(A) \to \boldsymbol{P}(A)$ such that for any $B \subseteq A$, $g(B) = A - B$

Examples of constants that do not satisfy (LT):

(27) The 1-place predicate "identical with **a**" (**a** is a singular term of the language), defined by $f_{=a}(\mathfrak{A}) = \{b : b \in A \;\&\; b = \mathbf{a}^{\mathfrak{A}}\}$, where $\mathbf{a}^{\mathfrak{A}}$ is the denotation of **a** in $\mathfrak{A}$

(28) The 1-place (predicative) quantifier "pebbles in the Red Sea," defined by $f_{\text{pebbles}\ldots}(\mathfrak{A}) = \{B : B \subseteq A \;\&\; B$ is a nonempty set of pebbles in the Red Sea$\}$

(29) The first-level membership relation (a 2-place first-level relation whose arguments are singular terms), defined by $f_{\in}(\mathfrak{A}) = \{\langle a, b\rangle : a, b \in A \;\&\; b$ is a set $\&\; a$ is a member of $b\}$

The definitions of these constants violate condition (E). To see why (29) fails, think of two models, $\mathfrak{A}$ and $\mathfrak{A}'$ with universes $\{0, \{0, 1\}\}$ and {Jean-Paul Sartre, Albert Camus} respectively. While the first-order structures $\langle\{0, \{0, 1\}\}, \langle 0, \{0, 1\}\rangle\rangle$ and ⟨{Jean-Paul Sartre, Albert Camus}, ⟨Jean-Paul Sartre, Albert Camus⟩⟩ are isomorphic (when taken as first-order, i.e., when the first elements are treated as sets of atomic objects), $\langle 0, \{0, 1\}\rangle \in f_{\in}(\mathfrak{A})$ but ⟨Jean-Paul Sartre, Albert Camus⟩ $\notin f_{\in}(\mathfrak{A}')$.

Another term that is not logical under (LT) is the definite-description operator ι. If we define ι (a quantifier functor) by a function f that, given a model $\mathfrak{A}$ with a universe A, assigns to $\mathfrak{A}$ a partial function h from $\boldsymbol{P}(A)$ into A, then condition (C.d) is violated. If we make h universal, using some convention to define the value of h for subsets of A that are not singletons, it has to be shown that the convention does not violate (E). We can, however, construct a 2-place predicative logical quantifier "the," which expresses Russell's contextual definition of the description operator:

(30) $f_{\text{the}}(\mathfrak{A}) = \{\langle B, C\rangle : B \subseteq C \subseteq A \;\&\; B$ is a singleton set$\}$

7 A New Conception of Logic

The definition of logical terms in section 6 gives new meaning to "first-order logic" based on Tarski's ideas. "First-order logic" is now a schematic title for any system of logic with a complete collection of truth-functional connectives and a nonempty set of logical constants. It is open to us, the users, to choose which particular set of constants satisfying (LT) we want to include in our first-order system. The logic itself is an open framework: any term may be plugged in as a logical constant, provided this is done in accordance with conditions (A) to (E). Any first- or second-level *formal* term is acceptable, so long as it is incorporated into the system "in the right way." The general framework of logic based on this conception I will call *Unrestricted Logic* or UL. I will also refer to it as *Tarskian Logic*, since it is based on Tarski's conception of the task and structure of logic. A particular system of Tarskian logic is simply *a logic*. Both syntactically and semantically the new logic preserves the form of definition characteristic of standard mathematical logic: syntactically, the logical constants serve as "formula-building operators" on the basis of which the set of well-formed formulas is defined by induction; semantically, the logical constants are associated with pre-fixed rules, to be used in the recursive definition of satisfaction in a model. Thus, for example, the syntactic definition of the 2-place quantifier "most" is given by the following clause:

- If Φ and Ψ are well-formed formulas, then $(\text{Most}^{1,1} x)(\Phi, \Psi)$ is a well-formed formula.

The rule associated with "most" is expressed in the corresponding semantic clause:

- If Φ and Ψ are well-formed formulas, $\mathfrak{A}$ is a model with a universe A, and g is an assignment of individuals in A to the variables of the language, then

$\mathfrak{A} \models (\text{Most}^{1,1} x)(\Phi, \Psi)[g]$ iff

$\langle\{a \in A : \mathfrak{A} \models \Phi[g(x/a)]\}, \{a \in A : \mathfrak{A} \models \Psi[g(x/a)]\}\rangle \in f_{M^{1,1}}(\mathfrak{A})$.

I will give a precise account of UL in chapter 4. In the meantime, I propose this provisional definition:

DEFINITION UL $\mathscr{L}$ is a logic in UL iff $\mathscr{L}$ is a Tarskian first-order system with (1) a complete set of truth-functional connectives and (2) a nonempty set of logical terms, other than those in (1), satisfying (LT).

I will now show (what should be clear from the foregoing discussion) that UL satisfies the pretheoretical requirements (C1) and (C2). Namely, if $\mathscr{L}$ is a first-order system in UL, then the Tarskian definition of "logical consequence" for $\mathscr{L}$ gives results in agreement with (C1) and (C2).

First the case for (C1). It suffices to show that the assumption (AS) (of section 2) holds for UL. Let $\mathscr{L}$ be any system of UL with new logical constants, let $\mathscr{C}$ be the logical vocabulary of $\mathscr{L}$, and let L be its extralogical vocabulary. The claim is that if Φ is a well-formed formula of $\mathscr{L}$, every possible extension of Φ relative to the vocabulary of $\mathscr{L}$ is represented by some model for $\mathscr{L}$ (where the extension of a sentence is taken to be a truth value, T or F).

I will sketch an outline of a proof. Suppose that Φ is an atomic formula of the form "Px," where P is an extralogical constant. The strong semantic variability of P and the other primitive terms in L ensures that every possible state of affairs relative to these terms is represented by some model $\mathfrak{A}$ for $\mathscr{L}$. So the claim holds for Φ. Now let Φ be of the form "$(Qx)\Psi x$," where Q is a quantifier and "Ψx" is (for the sake of simplicity) a formula with one free variable x. Assume the claim holds for "Ψx." Q, being a member of $\mathscr{C}$, is semantically rigid. Furthermore, its rigid interpretation is formal. But formal properties and relations intuitively do not change from one possible state of affairs to another. That is, while the number of, say, red things does vary among possible states of affairs, the second-level formal property "having n objects in X's extension" does not. Having n objects in a property's extension is always the same thing, no matter what property and what state of affairs we are considering. Therefore, the variability of situations with respect to "$(Qx)\Psi x$" is reduced to the variability of situations with respect to "Ψx." It is possible that "$(Qx)\Psi x$" has the extension T/F iff it is possible that "Ψx" has an extension representable by a subset B of the universe of some model $\mathfrak{A}$ such that $B \in f_Q(\mathfrak{A})/B \notin f_Q(\mathfrak{A})$. But by (the inductive) assumption, every possible extension of "Ψx" (relative to the vocabulary of $\mathscr{L}$) is represented by some model for $\mathscr{L}$. So if it is possible for "Ψx" to have an extension as required, there is a model that realizes this possibility. In this model the extension of "$(Qx)\Psi x$" is T/F. We can carry on this inductive reasoning with respect to any type of logical terms under (LT).

The case for (C2) is straightforward. Condition (E) expresses an intuitive notion of formality: to be formal is, intuitively, to take only structure into account. Within the scheme of model-theoretic semantics, to be formal is to be invariant under isomorphic structures. Now in UL, as in standard logic, logical consequences depend on the logical vocabulary of

the language. The formality of logical terms ensures that logical consequences do not rest on empirical evidence and do not distinguish the identity of objects in any given universe. Hence logical consequences of UL are formal in Tarski's sense.

Logics equivalent or similar to UL are often called in the literature "generalized logics," "extended logics," "abstract logics," or "model-theoretic logics." These labels may, however, convey the wrong message. Driving a wedge between "core" logic and its new "extensions," they seem to intimate that the "tight" and "lean" standard system is still the true logic. Such an interpretation of UL would, however, be wrongheaded. UL is not an abstract generalization of *real logic*. UL *is* real logic, full-fledged. As we have seen earlier in this chapter, the basic semantic principles of "core" logic (formulated by Tarski in the mid 1930s) are not fully materialized in the "standard" system. This system fails to produce all the formally necessary, i.e., "logical," consequences with a first-level vocabulary. It takes the full spectrum of UL logics to carry out the original program.

I have answered the question posed at the end of section 2. The broadest notion of logical term compatible with the intuitive concept of "logical consequence" is that of (LT). (LT) redefines the boundaries of logic, leading to the unrestricted system of UL. Condition (E) is especially important in determining the full scope of logic. It is worthwhile to trace the origins of this condition.

8 Invariance under Isomorphic Structures

The condition of invariance under isomorphic structures first appeared, as a characterization of logicality, in Lindenbaum and Tarski 1934–1935. Referring to a simple Russellian type-theoretic logic, Lindenbaum and Tarski proved a theorem that informally says, "Every relation between objects (individuals, classes, relations) which can be expressed by purely logical means [i.e., without using extralogical terms] is invariant with respect to every one-one mapping of the 'world' (i.e., the class of all individuals) onto itself."[20]

Now the metalanguage from which we draw the pool of logical terms is roughly equivalent to a subsystem of "pure" higher-order logic with Russellian simple types. For this language, Lindenbaum and Tarski's theorem shows that all definable notions satisfy the isomorphism condition with respect to "the world" (a "universal" model, in our terminology). The Lindenbaum-Tarski theorem appears to assume a notion of logicality that depends on the classification of the standard logical operators of a simple Russellian type theory as "purely logical." However, it follows from this very theorem that the standard operators themselves are invariant under isomorphic substructures, i.e., given any model 𝔄 (a submodel relative to Lindenbaum and Tarski's "universal" model) and a 1-place formula Φx, "$(\forall x)\Phi x$" is true in 𝔄 iff for any 1-place formula Ψx whose extension in 𝔄 is obtained from that of "Φx" by some permutation of the universe, "$(\forall x)\Psi x$" is true in 𝔄, and similarly for the other Russellian operators. So the theorem shows (relative to a simple type-theoretic language and the standard rules of logical proof) that Russellian logical terms and all terms that can be defined from them are "purely logical."

The idea that logical notions are distinguished by their invariance properties next appeared in Mautner's "An Extension of Klein's Erlanger Program: Logic as Invariant-Theory" (1946). Inspired by Klein's program of classifying geometrical notions in terms of invariance conditions, Mautner showed that standard mathematical logic can be construed as "invariant-theory of the symmetric group . . . of all permutations of the domain of individual variables."[21]

In his pioneering 1957 paper "On a Generalization of Quantifiers," Mostowski used the invariance property, for the first time, to license a genuine extension of standard first-order logic by adding new logical terms. Mostowski's condition technically was invariance under permutations of sets induced by permutations of the universe (of a given model). Informally, it was to be construed as the claim, (LQ2) of chapter 2, that quantifiers do not take into account the identity of individuals in the universe of discourse. Mostowski's criterion included references to the aforementioned papers of Lindenbaum and Tarski (1934–1935) and Mautner (1946).[22]

In 1966 Per Lindström generalized Mostowski's condition to full invariance under isomorphic (relational) structures, augmenting Mostowski's system with many-place predicative and relational quantifiers, often referred to as "Lindström quantifiers." There is a minor difference between Lindström's definition and (E) above: Lindström's structures are relational, and 0-place relations are not individuals but truth values, T or F. Thus mathematical structures involving individuals cannot be directly represented by logical terms, as in (24). Lindström, unlike Mostowski, was silent regarding the philosophical significance of his generalization. One might say his remarkable theorems solidify the distinguished status of standard first-order logic, but here again, it is unclear whether Lindström himself considers compactness and the Löwenheim-Skolem property to be

essential ingredients of logicality or mere mathematically interesting features of one among many genuinely logical systems. This philosophical disengagement is characteristic of the abundant literature on "abstract logic" that has followed Lindström's work.[23]

I often wondered what Tarski would have thought about the conception of Tarskian logic proposed in this book. After the early versions of the present chapter had been completed, I came upon a 1966 lecture by Tarski, first published in 1986, that delighted me in its conclusion. In the lecture "What are Logical Notions?" Tarski proposed a definition of "logical term" that is coextensional with condition (E):

> Consider the class of *all* one-one transformations of the space, or universe of discourse, or "world" onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have . . . notions, all of a very general character. I suggest that they are the logical notions, that we call a notion "logical" if it is invariant under all possible one-one transformations of the world onto itself.[24]

The difference between Tarski's 1966 lecture and the earlier Lindenbaum and Tarski paper is that here Tarski explicitly talks about the scope of logical terms for a first-order framework. (Indeed, in his introduction to the posthumously published lecture, J. Corcoran suggests that we see it as a sequel to Tarski's 1936 "On the Concept of Logical Consequence," in which the scope of logical terms was left as an open question.) It follows from the above definition, Tarski now says, that no term designating an individual is a logical term; the truth-functional connectives, standard quantifiers, and identity are logical terms; Mostowski's cardinality quantifiers are logical, and in general, all predicates definable in standard higher-order logic are logical. Tarski emphasizes that according to his definition, any mathematical property can be seen as logical when construed as higher-order. Thus, as a science of individuals, mathematics is different from logic, but as a science of higher-order structures, mathematics is logic.

The analysis that led to the extension of "logical term" in Tarski's lecture is, however, different from that proposed here. Tarski, like Mautner, introduced his conception as a generalization of Klein's classification of geometrical disciplines according to the transformations of space under which the geometrical concepts are invariant. Abstracting from Klein, Tarski characterized logic as the science of all notions invariant under one-to-one transformations of the universe of discourse ("space" in a generalized sense). My own conclusions, on the other hand, are based on analysis of Tarski's early work on the philosophical foundations of logic.

This is the reason that, unlike in the later Tarski, the criterion for logical terms proposed here includes, but is not exhausted by, condition (E). To be a logical term is not just to be a higher-level, mathematical term; it is to be incorporated in a certain syntactic-semantic system in a way that allows us to identify all intuitively logical consequences by means of a given rule, e.g., Tarski's (LC).

Following Lindström (Tarski's 1966 lecture remained unknown for a long time), condition (E) has been treated by mathematical logicians as a criterion for *abstract* logical terms. In the last decade condition (E), and some variants thereof, began to appear as a criterion of logicality in the formal semantic literature, often in combination with other criteria, like conservativity. If my analysis is correct, conservativity and other linguistic properties constraining (E) have nothing to do with logicality.

The only thorough philosophical discussion of condition (E) that I know of appears in Timothy McCarthy's 1981 paper "The Idea of a Logical Constant."[25] McCarthy rejects (E) as a sufficient condition for "logicality" on the grounds that it does not prevent the definition of logical terms by means of "contingent" expressions. To illustrate McCarthy's point, let us consider the quantifier "the number of planets," defined by

$$f_{\text{the number of planets}}(\mathfrak{A}) = \{B : B \subseteq A \ \& \ |B| = \text{the number of planets}\}.$$

Clearly, the quantifier "the number of planets" satisfies (E). Now

(31) The number of planets = 9

is contingent in the metalanguage, i.e, its extension changes from one "possible world" (in which we interpret the metalanguage) to another. Consider the sentence

(32) (The number of planets x)(Px & ~ Px).

This sentence is logically false as a matter of fact, McCarthy would say, that is, as a matter of the fact that the number of planets is larger than zero. However, in the counterfactual situation in which our sun had no satellites, (32) would turn out logically true. Therefore, "the number of planets x" will not do as a logical quantifier.

McCarthy's objection, however, does not affect my criterion, which includes conditions (A) to (D) in addition to (E). Condition (B) states that logical terms are identified with their (actual) extensions, so that in the metatheory the definitions of logical terms are rigid. Qua quantifiers, "the number of planets" and "9" are indistinguishable. Their (actual) extensions determine one and the same formal function over models, and this

function is a legitimate logical operator. In another world another description (and possibly another symbol) may designate this function. But that has no bearing on the issue in question. Inscription (32) may stand for different statements in different worlds. But the logical statement (32) is the same, and false, in all worlds. For that reason logic—Unrestricted Logic or any logic—is invariant across worlds. From the point of view of logic presented here, McCarthy's demand that the meaning of logical terms be known a priori is impertinent. The question is not how we come to know the meaning of a given linguistic expression, but how we set out to use it. If we set it up as a rigid designator of some formal property in accordance with conditions (A) to (E), it will work well as a logical constant in any Tarskian system of logic. Set differently, it might not. Switching perspectives, we may say that the only way to understand the meaning of a term used as a logical constant is to read it rigidly and formally, i.e., to identify it with the mathematical function that semantically defines it.

9 Conclusion

We have arrived at a general theory of the scope and nature of logical terms based on analysis of the function of logic and the philosophical guidelines at the basis of modern semantics. Given the breadth of the logical enterprise, we discovered that the standard terms alone do not provide an adequate superstructure. Yet in view of its goal, not every term can be used as a constant in Tarskian logic. There exists a clear, unequivocal criterion for eligible terms, and the terms satisfying this criterion far exceed those of "standard" logic.

We can now answer the questions posed at the end of chapter 2. Mostowski's claim that standard mathematical logic does not exhaust the scope of first-order logic has been vindicated. His semantic criterion on quantifiers, namely, "not distinguishing the identity of individuals in the universe," is most naturally interpreted as not discerning the difference between isomorphic structures. As for logicality and cardinality, the invariance condition implies that the two coincide in the case of predicative quantifiers, but in general, these notions are not essentially connected.

The next task is to outline a complete system of first-order logic with logical terms satisfying (LT). The series of conditions proposed in the present chapter constitute a definition of logical terms "from above": one can understand the conditions without thereby knowing how to construct all constants possessing the required properties. In the next chapter I will give a *constructive* definition of logical constants, inspired by Mostowski.

Mostowski's correlation of quantifiers with cardinality functions did to "predicative" generalized logic what the association of connectives with Boolean truth functions earlier did to sentential logic. It provided a highly informative answer to the questions, "What is a predicative quantifier?" "What are all the predicative quantifiers?" Following Mostowski, I will present a correlation of logical terms with mathematical functions of a certain kind so that the totality of functions will determine the totality of logical terms and each function will embed the "instructions" for constructing one logical term from the total list.

Chapter 4
Semantics from the Ground Up

Our philosophical analysis in the last chapter has led to the conclusion that any second-level mathematical predicate can be construed as a logical quantifier under a semantic definition satisfying the metatheoretical conditions (A) to (E). Since the predicative quantifiers defined in chapter 2 satisfy these conditions, they are genuine logical quantifiers, and Mostowski's claim that they belong in a systematic presentation of symbolic logic is justified. Our analysis also provides an answer to the question "Which second-level predicates on relations are logical quantifiers?" Relational quantifiers are simply logical terms of a particular type: second-level predicates or relations whose arguments include at least one first-level relation (many-place predicate).

On my analysis, Mostowski's semantic condition on predicative quantifiers, (LQ2), the requirement that quantifiers should not distinguish the identity of elements in the universe of a given model, corresponds to Tarski's (C2), the requirement that logical terms (and hence logical quantifiers) be *formal*. Like Mostowski, I interpret (C2) as an invariance condition, and this condition, when applied to predicative quantifiers, coincides with his. More accurately, Mostowski's rendering of (LQ2) as invariance under permutations of sets induced by permutations of the universe is generalized to condition (E), which says that logical terms in general are invariant under isomorphic structures. In terms of Mostowski's definition of quantifiers as functions from sets to truth values, I say that a logical term over universe A is a function q from sequences of relations (predicates, individuals) of the right type to truth values, T or F, such that if s is a sequence in $\mathrm{Dom}(q)$ and m is a permutation of A,

$$q(s) = T \text{ iff } q(m(s)) = T,$$

where $m(s)$ is the image of s under m.

The characterization of logical constants in terms of invariance under permutations of the universe is still not very informative, however. In the case of predicative quantifiers, Mostowski was able to establish a one-to-one correspondence between quantifiers satisfying (LQ2) and cardinality functions of a specified kind, and this resulted in a highly informative characterization of predicative quantifiers: predicative quantifiers attribute *cardinality* properties (relative to the cardinality of a given universe) to the extensions of 1-place first-level predicates in their scope; the functions t associated with predicative quantifiers constitute "rules" for constructing predicative quantifiers over a universe A. Although cardinality functions can be extended to logical terms other than predicative quantifiers, they evidently will not cover all the logical terms over a universe A. The latter express *structural* properties of sets, relations, and individuals in general, not just cardinality properties.

My main goal in the present chapter is to develop a semantic definition of logical terms that captures the idea of *formal structure* in a way analogous to that in which Mostowski's definition captures the idea of cardinality. Mostowski's definition distinguishes sets according to their size relative to the size of a given universe. I want to characterize all formal patterns of individuals standing in relations within an arbitrary universe A and then distinguish relations according to the formal patterns they exhibit. This will be the basis for my "constructive" definition of logical terms over A. But first I will examine the original characterization of logical terms satisfying (E), due to Per Lindström.

1 Lindström's Definition of "Generalized Quantifiers"

In "First Order Predicate Logic with Generalized Quantifiers" Lindström (1966a) associates generalized quantifiers with classes of structures (models) closed under isomorphism. More precisely, his semantic definition goes as follows:

DEFINITION LQ A quantifier is (semantically) a class $\bar{Q}$ of relational structures of a single type $t \in \omega^n$, $n > 0$, closed under isomorphism,

where a relational structure is a sequence consisting of a universe (a set) and a series of constant relations on, or subsets of, the universe (but not individuals). The type of structure $\mathfrak{A}$ is an ordered n-tuple, $\langle m_1, \ldots, m_n \rangle$, where n is the number of constant relations R_i in $\mathfrak{A}$ and m_i, $1 \leq i \leq n$, is the number of arguments of the relation R_i. (A truth value is considered by Lindström a relation with no arguments. There are only two

0-place relations, T and F.) Each semantic quantifier $\bar{Q}$ is symbolized by a syntactic quantifier Q; different syntactic quantifiers corresponding to different semantic quantifiers. If Q symbolizes $\bar{Q}$, Q is said to be of the type t common to all the structures in $\bar{Q}$. A syntactic quantifier Q of type $t = \langle m_1, \ldots, m_n \rangle$ is a quantifier in $m_1 + m_2 + \cdots + m_n$ variables that attaches to n formulas to form a new formula.

The truth conditions for formulas with Lindström quantifiers are defined as follows: Let Q be a Lindström quantifier of type $t = \langle m_1, \ldots, m_n \rangle$. Let $\Phi_1, \ldots, \Phi_n$ be formulas of first-order logic with Lindström quantifiers. Let $\bar{x}_1, \ldots, \bar{x}_n$ be a series of n pairwise disjoint elements, where for $1 \leq i \leq n$, the element $\bar{x}_i$ is a series of m_i distinct variables. Let $\mathfrak{A}$ be a model with universe A, and let g be an assignment of elements in A to the individual variables of the language. Then

$\mathfrak{A} \models (Q\bar{x}_1, \ldots, \bar{x}_n)(\Phi_1, \ldots, \Phi_n)[g]$ iff the structure

$\langle A, \Phi_1^{\mathfrak{A}}\bar{x}_1[g], \ldots, \Phi_n^{\mathfrak{A}}\bar{x}_n[g] \rangle$ is a member of $\bar{Q}$,

where for $1 \leq i \leq n$,

$$\Phi_i^{\mathfrak{A}}\bar{x}_i[g] = \begin{cases} \mathrm{T} & \text{if } \bar{x}_i = \langle\ \rangle \text{ and } \mathfrak{A} \models \Phi_i[g] \\ \mathrm{F} & \text{if } \bar{x}_i = \langle\ \rangle \text{ and } \mathfrak{A} \not\models \Phi_i[g] \\ \{\bar{a}_i : \mathfrak{A} \models \Phi_i[g(\bar{x}_i/\bar{a}_i)]\} & \text{otherwise} \end{cases}$$

("$\bar{a}_i$" stands for an arbitrary sequence of m_i elements of A, $a_{i_1}, \ldots, a_{i_{m_i}}$, and "$g(\bar{x}_i/\bar{a}_i)$" abbreviates "$g(x_{i_1}/a_{i_1}) \ldots (x_{i_{m_i}}/a_{i_{m_i}})$").

Clearly, the quantifiers definable in Lindström's logic include all the logical quantifiers of chapter 3 over (sequences of) predicates and relations (but not over sequences including individuals). In addition, all the logical predicates and all the truth-functional connectives are definable as Lindström quantifiers. Thus we have the following:

(1) The existential quantifier of standard logic is defined as $\bar{E}$ = the class of all structures $\langle A, P \rangle$, where A is a set, $P \subseteq A$, and P is not empty.

(2) The predicative quantifier R^2 of chapter 2 ("there are more . . . than - - -") is defined as $\bar{R}^2$ = the class of all structures $\langle A, P_1, P_2 \rangle$, where A is a set, $P_1, P_2 \subseteq A$, and $|P_1| > |P_2|$.

(3) The "well-ordering" relational quantifier of chapter 3, WO, is defined as $\bar{WO}$ = the class of all structures $\langle A, R \rangle$, where A is a set, $R \subseteq A^2$, and R well-orders Fld(R).

(4) The negation of sentential logic is defined as $\bar{N}$ = the class of all structures $\langle A, F \rangle$, where A is a set. (The structure $\langle A, F \rangle$ is non-isomorphic to $\langle A, T \rangle$ by definition.)

(5) The disjunction of sentential logic is defined as $\dot{D}$ = the class of all structures $\langle A, S_1, S_2 \rangle$, where A is a set and S_1, S_2 are truth values, at least one of which is T.

My definition of logical terms in chapter 3 essentially coincides with Lindström's. There are some small differences in the construction of models: Lindström's models include the two truth values T and F as components. This allows him to construe the truth-functional connectives as logical quantifiers. (Indeed, I could incorporate the same device in my theory.) In addition, Lindström does not consider structures with individuals. It is easy, however, to extend his definition to structures of this kind, and given such an extension, all logical terms of (LT) will fall under Lindström's definition. There is also a minor difference between Lindström's syntax and mine: whereas I constructed an n-place predicative quantifier as binding a single individual variable in any n-tuple of well-formed formulas in its domain, Lindström's predicative quantifiers bind n distinct variables. Thus what I symbolize as

$(Qx)(\Phi_1 x, \ldots, \Phi_n x)$

Lindström symbolizes as

$(Qx_1, \ldots, x_n)(\Phi_1 x_1, \ldots, \Phi_n x_n)$.

However, since the two quantifications express exactly the same statement, the difference just amounts to a simplification of the notation.

In chapter 1, I pointed out that the apparatus of Tarskian model-theoretic semantics is "too rich" for standard first-order logic. We never use the model-theoretic apparatus in its entirety to state the truth conditions of sentences of standard logic, to determine standard logical truths and consequences, to distinguish semantically between nonequivalent standard theories, etc. In particular, the collection of infinite models is to a large extent redundant because any sentence or theory represented by an infinite model is represented by uncountably many distinct infinite models (the Löwenheim-Skolem-Tarski theorem). The new conception of logic, which received its first full-scale expression in Lindström, enriches the expressive power of the first-order language so that the model-theoretic apparatus is put to full use. The extended logical vocabulary allows the formation of new sentences and theories, so evey model becomes the unique representation (up to isomorphism) of some theory of the new language. Put otherwise, every structure, up to isomorphism, is describable by a theory of the generalized language, indeed, in Lindström's system, by a single sentence (if the language has enough nonlogical constants of the "right" type). Thus, let $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ be a structure of type $t = \langle m_1, \ldots, m_n \rangle$. Let $\bar{Q}$ be the class of all structures $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, and let Q be the quantifier defined by $\bar{Q}$. Let $P_1, \ldots, P_n$ be distinct relational constants of $m_1, \ldots, m_n$ places respectively (P_i being a sentential letter if $m_i = 0$), and let $\bar{x}_1, \ldots, \bar{x}_n$ be series of distinct variables as explained above. Then the sentence

(6) $(Q\bar{x}_1, \ldots, \bar{x}_n)(P_1\bar{x}_1, \ldots, P_n\bar{x}_n)$

describes the unique structure $\mathfrak{A}$ (up to isomorphism).

Lindström's definition, however, is "from above." As such, it does not show us how to "construct" logical terms over a model $\mathfrak{A}$ using elements in the universe of $\mathfrak{A}$ as the initial building blocks. In addition, Lindström's definition of logical terms over a specific model $\mathfrak{A}$ involves quantification (in the metalanguage) over all models. Thus, to determine whether an n-tuple of formulas $\langle \Phi_1, \ldots, \Phi_n \rangle$ satisfies a quantifier Q in $\mathfrak{A}$, we need information not only on the extentions of $\Phi_1, \ldots, \Phi_n$ in $\mathfrak{A}$ but also about the class of all models for the language. In the next section I will propose a definition of logical terms "from the ground up." This definition shows how to build logical terms over $\mathfrak{A}$ out of constructs of elements of $\mathfrak{A}$ and without reference to the totality of models.

2 Constructive Definition of Logical Terms

The idea is this: Tarskian logical terms over a model $\mathfrak{A}$ with universe A distinguish the *form* or *structure* of sets, relations, and functions over A. Any two relations differing in structure will be distinguished by a logical term on A, but relations that share the same structure will not. Similarly for sets and functions. So, to define the totality of logical terms on A, we first have to define the totality of "structures" over A. Once we determine the totality of, say, structures of binary first-level relations over A, we can define 1-place binary relational quantifiers on A as functions that assign the value T to some of these structures but not to others (allowing, of course, for the two extreme cases of functions that assign the value T to all binary relational structures, and to none). The totality of these functions is the totality of binary relational quantifiers on A. The definition will be general enough to include all types of logical terms. For the sake of simplicity I will, however, omit logical functors and logical quantifier functors. It is easy to extend the definition to these logical terms as well.

Before I begin the formal presentation, I will explain the idea behind the definition by reference to a simple example.

An informal account

Suppose we have a universe with ten individuals, say Alan, Becky, Carl, Debra, Eddy, Fred, Gary, Helen, Ian, and Jane. We want to identify all structures involving these persons that are the extensions of (legitimate) first-order logical terms over a model $\mathfrak{A}$ with the above group as its universe. I will refer to this universe simply as "The Group."

Let us consider several structures involving members of the Group (designated by their initials):

(7) j

(8) $\{a, c, d, i\}$

(9) $\{a, b, c, d, e, f, g, h, i, j\}$

(10) $\{\{a, c, d, i\}\}$

(11) $\{\{a, b, c, d, e, f, g, h, i, j\}\}$

(12) $\{\{a\}, \{c\}, \{d\}, \{h\}\}$

(13) $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\}\}$

(14) $\{\langle a, a\rangle, \langle f, f\rangle, \langle g, g\rangle, \langle j, j\rangle\}$

(15) $\{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle e, e\rangle, \langle f, f\rangle, \langle g, g\rangle, \langle h, h\rangle,$
$\langle i, i\rangle, \langle j, j\rangle\}$

(16) $\{\varnothing, \{\langle a, j\rangle\}, \{\langle a, j\rangle, \langle c, d\rangle, \langle i, h\rangle\}, \{\langle a, j\rangle, \langle c, h\rangle, \langle g, d\rangle\}\}$

(17) $\{\{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}, \cdots \{\langle j, a\rangle, \langle a, b\rangle, \langle j, b\rangle\},$
$\{\langle a, b\rangle, \langle b, d\rangle, \langle a, d\rangle\}, \cdots \{\langle j, a\rangle, \langle a, c\rangle, \langle j, c\rangle\},$
$\{\langle a, b\rangle, \langle b, e\rangle, \langle a, e\rangle\}, \cdots \{\langle j, a\rangle, \langle a, d\rangle, \langle j, d\rangle\},$
$\vdots \quad \vdots \quad \vdots$
$\{\langle a, j\rangle, \langle j, g\rangle, \langle a, g\rangle\}, \cdots \{\langle j, i\rangle, \langle i, f\rangle, \langle j, f\rangle\},$
$\{\langle a, j\rangle, \langle j, h\rangle, \langle a, h\rangle\}, \cdots \{\langle j, i\rangle, \langle i, g\rangle, \langle j, g\rangle\},$
$\{\langle a, j\rangle, \langle j, i\rangle, \langle a, i\rangle\}, \cdots \{\langle j, i\rangle, \langle i, h\rangle, \langle j, h\rangle\}\}$

(18) $\{\varnothing, \langle\{\langle a, j\rangle\}, b\rangle, \langle\{\langle c, d\rangle, \langle i, h\rangle\}, e\rangle,$
$\langle\{\langle c, h\rangle, \langle g, d\rangle\}, f\rangle\}$

(19) $\{\langle\{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}, a\rangle, \cdots \langle\{\langle j, a\rangle, \langle a, b\rangle, \langle j, b\rangle\}, j\rangle,$
$\langle\{\langle a, b\rangle, \langle b, d\rangle, \langle a, d\rangle\}, a\rangle, \cdots \langle\{\langle j, a\rangle, \langle a, c\rangle, \langle j, c\rangle\}, j\rangle,$
$\langle\{\langle a, b\rangle, \langle b, e\rangle, \langle a, e\rangle\}, a\rangle, \cdots \langle\{\langle j, a\rangle, \langle a, d\rangle, \langle j, d\rangle\}, j\rangle,$
$\vdots \quad \vdots \quad \vdots$
$\langle\{\langle a, j\rangle, \langle j, g\rangle, \langle a, g\rangle\}, a\rangle, \cdots \langle\{\langle j, i\rangle, \langle i, f\rangle, \langle j, f\rangle\}, j\rangle,$
$\langle\{\langle a, j\rangle, \langle j, h\rangle, \langle a, h\rangle\}, a\rangle, \cdots \langle\{\langle j, i\rangle, \langle i, g\rangle, \langle j, g\rangle\}, j\rangle,$
$\langle\{\langle a, j\rangle, \langle j, i\rangle, \langle a, i\rangle\}, a\rangle, \cdots \langle\{\langle j, i\rangle, \langle i, h\rangle, \langle j, h\rangle\}, j\rangle\}$

How shall we decide which of these structures are the extensions of logical terms over a model $\mathfrak{A}$ with the Group as its universe? The answer follows directly from the criterion for logical terms in chapter 3: a structure is the extension of a legitimate logical term iff it is closed under permutations of the universe. I will call such a structure a *logical structure*. Thus if S is a logical structure that contains the element E, then S also contains every element E' that can be obtained from E by some permutation of the universe. Let us examine each of the above structures and see what kind of structure it is.

Structure (7) consists of a particular member of the Group, Jane. Jane is not preserved under permutations of the Group, because such permutations may assign Fred to Jane, and Fred is not Jane. Jane (like Fred, Ian, and the rest) is not a "logical individual." Indeed, it is a basic principle of logic that there are no logical individuals and individuals do not constitute the extension of any logical term.

Structure (8) is also not closed under permutations of the universe. A permutation that assigns Jane to Alan, Alan to Carl, Helen to Debra and Gary to Ian, will carry us beyond $\{a, c, d, i\}$ to $\{a, g, h, j\}$. Here (8) may be the extension of the first-level predicate "x is redheaded," or "x is a leftist." But (8) does not represent any first-level logical property of members of the Group.

Structure (9), on the other hand, does represent a first-level logical property, since (9) is preserved under all permutations of the universe. Thus no matter who is assigned to Jane by a given permutation m, this person is already in (9). Put differently, the universal set is its own image under all permutations of the universe. We can associate with this set the property of being a member of the Group or see it as the property of being American, etc. No matter what other properties are "extentiated" in the Group by the universal set, (9) is also an instantiation of the logical property of self-identity over the Group and hence is a logical structure.

Structure (10), like (8), is not logical. It may be the extension of the second-level predicate "P is a property of redheads," or "P is an attribute of leftists." But these do not coincide with any second-level logical properties of members of the Group.

Structure (11), however, is the extension of a logical term, namely the universal quantifier over the Group.

Structure (12) is also nonlogical, since it is not closed under permutations of the universe. Suppose that among the members of the Group Alan is the only philosopher, Helen is the only linguist, Carl is the only historian, and Debra is the only novelist. Then (12) may be the extension of the nonlogical second-level predicate "*P* is either a distinctive characteristic of philosophers, a distinctive characteristic of linguists, a distinctive characteristic of historians, or a distinctive characteristic of novelists." But (12) cannot be the extension of any logical term over the Group.

Structure (13), unlike (12), is logical. Structure (13) is the extension of the quantifier "there is exactly one *x* such that" over 𝔄. As a predicate, (13) is the second-level attribute "*P* is a property of exactly one individual," an attribute whose extension is invariant under permutations of the Group.

Structure (14) too is nonlogical. Structure (14) may be the extension of "*x* likes *y*'s dog(s)" over the Group (each dog owner likes his own dog(s)), or it may be the extension of some other relation over the Group, but the relation in question is not logical, and (14) cannot exhaust the extension of any logical term over the Group.

Structure (15) is the familiar relation of identity. This relation is closed under permutation of the universe and hence is logical.

Structure (16) may be the extension of the second-level predicate "*X* is the set of married pairs (husband and wife) in 1981, or *X* is the set of married pairs in 1982, or . . . , or *X* is the set of married pairs in 1990." Thus (16) reflects the various matrimonial constellations within the Group in the last decade. For example, during the first five years there were no marriages among members of the Group. Then in 1986 Alan married Jane, in 1987 Carl married Debra and Ian married Helen, and in 1989 Debra divorced Carl and married Gary, while Carl married Helen, who divorced Ian. This chronicle is clearly not closed under permutations of members of the Group.

Structure (17), on the other hand, is closed under permutations. It represents a linear ordering of triples in general. Structure (17) makes up the extension of the relational quantifier "*R* is a strict linear ordering of triples." This quantifier, symbolized by Q, will appear in formulas of the form "(Q*xy*)Φ." Thus if three members of the Group graduated from Columbia College, and their graduation dates do not coincide, the statement "(Q*xy*) *x* graduated from Columbia College before *y*" will turn out true in the universe in question.

Another nonlogical structure is given by (18). Suppose that there are three children in the Group: Becky, born to Alan and Jane in 1986, Eddy, born to Carl and Debra in 1987, and Fred, born to Gary and Debra in 1989. A second-level predicate that records births in the Group next to weddings (of men to women, by year, as in (16)), may have (18) as its extension.

Finally, (19) is a logical structure of pairs consisting of a strict linear ordering of a triple and its smallest element. This structure "extentiates" a relational quantifier over pairs of a binary relation and an individual, similar to (24) of chapter 3.

The principle of closure under permutations determines all the logical terms over a given universe. Every structure containing sets of individuals, relations of individuals, sequences of these, or sequences of sets/relations and individuals and closed under permutations of the universe determines a legitimate logical term over that universe. But the principle of closure under permutations can be used not only to identify but also to construct logical structures over a universe *A*. The construction of such structures is a very simple matter.

Again, take the Group. Construct any set of members of the Group, say $\{a, b, d, f\}$. Examine all permutations *m* of members of the Group and for each such permutation *m* add $m(a)$, $m(b)$, $m(d)$ and $m(f)$ to your set. In other words, close the set $\{a, b, d, f\}$ under all permutations of the universe, or create a union of all its images under such permutations. You will end up extending $\{a, b, d, f\}$ to (9), the universal set of the domain. This set is the extension of the first-level logical predicate of self-identity over the Group.

In a similar manner you can start from the relation (14), and by uniting (creating a union of) all its images under permutations of the universe, you will obtain the logical structure (15), the extension of the binary logical relation of identity.

Likewise, (17) can be obtained from $\{\{\langle a, b\rangle, \langle b, j\rangle, \langle a, j\rangle\}\}$ by closing it under permutations. And so on.

Suppose now you start with $\{\varnothing, \{a\}, \{a, b\}\}$. Closure under permutations will give you a set whose members are the empty set, all unit sets, and all sets of two elements. This set is the extension of the 1-place predicative quantifier "there are at most two" over the Group.

I have characterized the logical terms over a single universe, but my theory of logical terms says that logical terms do not distinguish between universes of the same cardinality. That is, each logical term is defined by a rule that does not change from one universe of cardinality α to another. Thus, although the characterization of identity for the Group by (15) would do, this is evidently not an adequate characterization for all universes with 10 elements. To capture the idea of a logical term, the *rule* associated with such a term, rather than its extension in a particular universe, should be specified. A very simple method of associating terms with rules presents itself. The idea is this: instead of recording the actual extension of a given term in a given universe, let us record its "index extension." Unlike its "object extension," the index extension encodes a rule that applies to all universes of the same cardinality. We can then distinguish between rules that do, and rules that do not, correspond to logical terms over universes of the cardinality in question.

I will begin by specifying a fixed index set for all universes of a given cardinality. In case of the Group, I will take 10, identified with the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, as my index set. More generally, if A is a universe of cardinality α, I will take the least ordinal of cardinality α, defined as the set of all smaller ordinals, to be a standard index set for all universes of cardinality α. I will say that *A is indexed by* α or, in the example above, that the Group is indexed by 10. There are, of course, many ways of indexing the Group by 10. We may start any way we want, say assigning 0 to Alan, 1 to Becky, and so on, following the alphabetical order of the members's first names. Next we associate with each structure generated from members of the Group its index image under the chosen indexing. Thus the index image of (14) is

(20) {⟨0, 0⟩, ⟨5, 5⟩, ⟨6, 6⟩, ⟨9, 9⟩}.

The index image of (15) is

(21) {⟨0, 0⟩, ⟨1, 1⟩, ⟨2, 2⟩, ⟨3, 3⟩, ⟨4, 4⟩, ⟨5, 5⟩, ⟨6, 6⟩, ⟨7, 7⟩, ⟨8, 8⟩, ⟨9, 9⟩}.

And the index images of (7), (9), (11), and (16) are respectively

(22) 9,

(23) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9},

(24) {{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}},

(25) {∅, {⟨0, 9⟩}, {⟨0, 9⟩, ⟨2, 3⟩, ⟨8, 7⟩}, {⟨0, 9⟩, ⟨2, 7⟩, ⟨6, 3⟩}}.

Note that it is essential that we do not treat the members of 10 in the same way that we treat 10, namely as sets of all smaller numbers. The reason is that if we identify 9 with {0, 1, 2, . . . , 8}, (22) will represent not only (7) but also

(26) {*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*}.

Similarly, if we identify 0 with ∅, (25) will not distinguish between (16) and

(27) {*a*, {⟨*a*, *j*⟩}, {⟨*a*, *j*⟩, ⟨*c*, *d*⟩, ⟨*i*, *h*⟩}, {⟨*a*, *j*⟩, ⟨*c*, *h*⟩, ⟨*g*, *d*⟩}}.

Therefore, I define an *index set* to be a set of ordinals treated as individuals (or as sets of pairs of the form (β, a), where a is some fixed object). More precisely, an index set for a universe of cardinality α is the set of all ordinals smaller than the least ordinal of cardinality α, where the ordinals in the index set are themselves not sets of ordinals.

Back to the index set 10. I call a member of 10 a *10-individual*, a subset of 10 a *10-predicate*, and a set of n-tuples of members of 10 ($n > 1$) a *10-relation*. Thus (22) is a 10-individual, (23) is a 10-predicate, and (20) and (21) are 10-relations.

I call any finite sequence of 10-individuals, 10-predicates, and/or 10-relations a *10-argument*. Such sequences constitute the arguments of logical terms over the Group. It follows that a 10-individual is a *10-predicate-argument*; a finite sequence of two or more 10-individuals is a *10-relation-argument*; other 10-arguments are *10-quantifier-arguments*. I say that 10-arguments are of the same type if they have the same structure: all individuals are of the same type, all sets of individuals are of the same type, and all n-place relations of individuals are of the same type. Sequences of m elements of corresponding types are also of the same type. (The formal definition of type is slightly different, but the notion of "same type" is the same.) Thus

(28) ⟨1, 2⟩

and

(29) ⟨3, 4⟩

are of the same type, and so are

(30) {8}

and

(31) {3, 4, 5, 8},

as well as

(32) ⟨1, {1, 2}, {⟨1, 3⟩}⟩

and

(33) ⟨9, {3, 4, 5}, {⟨6, 7⟩, ⟨7, 6⟩}⟩.

I call two 10-arguments *similar* iff one is the image of the other under some permutation of 10. Thus (28) and (29) are similar, but neither (30) and (31) nor (32) and (33) are. Looking at the logical structures among (7) through (19), we see that a logical structure is a structure of similar elements of a given type. More accurately, a logical structure over the Group is a structure of 10-arguments of a single type closed under the relation of similarity. Since the relation of similarity is an equivalence relation, each logical structure corresponds to a union of equivalence classes of similar 10-arguments of a given type.

Note that while some logical terms can be identified with a single equivalence class, others correspond to a union of equivalence classes. For example, "there is *exactly* one" is a function that gives to a 10-argument the value T iff it is a member of the equivalence class of all sets similar to {0}, [{0}], but "there is *at least* one" assigns the value T to members of more than one equivalence class. So I define a logical term over universes with 10 elements as a function from all equivalence classes of a given type to {T, F}. "There is exactly one" assigns the value T to the equivalence class [{0}] and F to all other equivalence classes of subsets of the universe, whereas "there is at least one" assigns the value T to [{0}], [{0, 1}], ..., [{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}] and F to [∅]. I call such functions *10-operators*.

What can we do with 10-operators? Well, there are several things we can do. We can take a 10-operator of type t (that is, an operator defined over equivalence classes of elements of type t), a structure of the same type generated from the Group (a 10-individual being matched with a member of the Group, etc.), and ask whether the latter satisfies the logical term defined by the former. For example, we can take the extension of the predicate "x is a philosopher," namely $\{a\}$, and ask whether it satisfies a given 1-place predicative quantifier over the Group. To find the answer, we first index the Group by 10 (in any way we choose). Then we take the index image of $\{a\}$ and see whether the quantifier in question (defined as a 10-operator) gives the value T to [{index(a)}]. This test will show that

(34) (At least one x) x is a philosopher

is true in the intended model of the Group (Alan is a philosopher), but

(35) (!2 x) x is a philosopher

and

(36) $(\forall x)$ x is a philosopher

are false in the same model (Alan is the only philosopher).

Second, we can take a structure over the Group, ask whether it defines a logical term over the Group, and, if the answer is positive, get a general semantic schema of the logical term in question. We do this by creating an index image of the structure and examining whether the result is a union of equivalence classes under the relation of similarity. Thus (21), an index image of (15), is an equivalence class of all pairs similar to ⟨0, 0⟩ and therefore (15) does determine a logical term, namely identity, over the Group. The index image (20) of (14) does not constitute such an equivalence class (or a union of equivalence classes under similarity), and hence (14) does not determine a logical term over the Group.

Third, we can take any 10-operator and use it as a blueprint for constructing a logical term over the Group. Thus, starting with any indexing of the Group by 10, I take the 10-operator "exactly one," a function o from all equivalence classes of subsets of 10 to {T, F} defined as

$$o[N] = \mathrm{T} \text{ iff } [N] = [\{0\}],$$

and transform it into a quantifier in extension by going through the elements of the equivalence class(es) assigned T and constructing their correlates over the Group: $\{a\}$, $\{b\}$, etc. I then collect these correlates into a set, and this is (13), the quantifier "there is exactly one" over the Group.

Finally, I define the totality of all logical terms over the Group as the totality of predicates, relations, and quantifiers corresponding to all distinct 10-operators. Generalizing, I define the totality of logical terms as functions that to each cardinal α assign an α-operator.

A formal account

First, let me make some preliminary remarks. In the foregoing definitions I use the variable α to range over cardinals identified with equipollent least ordinals. But while I take a cardinal α to be a set of ordinals, I require that the ordinals in α are themselves not sets of ordinals. This requirement is introduced to ensure that "the index image of x," defined below, is one-to-one. (We can treat ordinals as individuals, we can replace each von Neumann ordinal α with the pair (β, a), where a is some fixed object, etc.) Throughout the book I use lowercase Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ both as variables ranging over cardinals and as variables ranging over ordinals. It is always clear from the context what the range of a given variable is.

I identify a 1-tuple with its member, i.e., $\langle x \rangle = x$.

In earlier chapters I often distinguished between predicates (1 place) and relations (many places). Below I will talk only about predicates, referring to relations as many-place predicates.

DEFINITION 1 Let A be a set indexed by $\alpha = |A|$, where an indexing of A by α is a one-to-one function from α onto A. The *index image* of x, $i(x)$, under the given indexing is as follows:

- If $x \in A$, $i(x) = (\iota\beta \in \alpha)(x = a_\beta)$.
- If $x \subseteq A^n$, $n \geq 1$, $i(x) = \{\langle \beta_1, \ldots, \beta_n \rangle \in \alpha^n : \langle a_{\beta_1}, \ldots, a_{\beta_n} \rangle \in x\}$.

TERMINOLOGY Let α be a cardinal number. An *α-individual* is a member of α; an n-place *α-predicate* is a subset of α^n.

If A is indexed by $\alpha = |A|$, then since the indexing function is one-to-one and onto, an α-individual is the index image of some $a \in A$, and an n-place α-predicate is the index image of some $R \subseteq A^n$, under the given indexing.

DEFINITION 2 Let k be a positive integer. I call $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ an *α-predicate-argument* if each $r_i(\alpha)$, $1 \leq i \leq k$, is an α-individual. I call $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ an *α-quantifier-argument* if each $r_i(\alpha)$, $1 \leq i \leq k$, is either an α-individual or an α-predicate and at least one $r_i(\alpha)$, $1 \leq i \leq k$, is an α-predicate. If $R(\alpha)$ is either an α-predicate-argument or an α-quantifier-argument, I say that $R(\alpha)$ is an *α-argument*.

Below I categorize various kinds of entities into "types." To simplify the type notation, I use two systems of categorization. Entities categorized by the first system will be said to have *marks*, and entities categorized by the second system will be said to have *types*. An entity with a type is a function, and its *type* is essentially the *mark* (sequence of *marks*) of its argument(s).

DEFINITION 3 A *type* is a sequence of natural numbers, $\langle t_1, \ldots, t_k\rangle$, $k > 0$. A *mark* is also a sequence of natural numbers, $\langle m_1, \ldots, m_k\rangle$, $k > 0$.

CONVENTION If p is the k-tuple

$$\underbrace{\langle 0, \ldots, 0\rangle}_{k \text{ times}},$$

I say that $p = 0^k$. If $p = 0^1$, I say that $p = 0$.

DEFINITION 4 Let $R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle$ be an α-argument. The *mark* of $R(\alpha)$, $m_R(\alpha)$, is a k-tuple, $\langle m_1, \ldots, m_k\rangle$, where for $1 \leq i \leq k$,

$$m_i = \begin{cases} 0 & \text{if } r_i(\alpha) \text{ is an } \alpha\text{-individual,} \\ n & \text{if } r_i(\alpha) \text{ is an } n\text{-place } \alpha\text{-predicate.} \end{cases}$$

DEFINITION 5 Let $R_1(\alpha)$, $R_2(\alpha)$ be two α-arguments. $R_1(\alpha)$ and $R_2(\alpha)$ are *similar* iff for some permutation m of α, $R_1(\alpha) = m(R_2(\alpha))$, where $m(R_2(\alpha))$ is the image of $R_2(\alpha)$ under the map induced by m (which I also symbolize by m).

TERMINOLOGY If $R(\alpha)$ is an α-argument, I designate the equivalence class of $R(\alpha)$ under the relation of similarity, defined above, as $[R(\alpha)]$. I call

$[R(\alpha)]$ a *generalized α-argument*. If $R(\alpha)$ is of mark p, I say that $[R(\alpha)]$ is also of mark p. I call a set of generalized α-arguments an *α-structure*.

DEFINITION 6 Let $[\mathfrak{R}(\alpha)]$ be the set of all generalized α-arguments of a given mark. An *α-operator* is a function

$$o_\alpha : [\mathfrak{R}(\alpha)] \to \{T, F\}.$$

If $[\mathfrak{R}(\alpha)]$ is a set of generalized α-predicate-arguments, I call o_α an *α-predicate-operator*; if $[\mathfrak{R}(\alpha)]$ is a set of generalized α-quantifier-arguments, I call o_α an *α-quantifier*. If the members of $[\mathfrak{R}(\alpha)]$ are of mark p, I say that o_α is of *type* p. We can identify an α-operator with an α-structure, namely the set of all $[R(\alpha)]$'s in its domain such that $o([R(\alpha)]) = T$.

To prove one-to-one correspondence between α-operators and logical predicates and quantifiers of UL restricted to $\mathfrak{A}(|\mathfrak{A}| = \alpha)$, we need a few additional definitions.

DEFINITION 7 If C is a logical predicate or quantifier satisfying conditions (A) to (E) of chapter 3, then *the restriction of C to $\mathfrak{A}$*, $C_{\mathfrak{A}}$, is as follows: Let $f_c(\mathfrak{A})$ be as in chapter 3, section 6. If $f_c(\mathfrak{A})$ is a subset of $B_1 \times \cdots \times B_k$ (see condition (C)), then $C_{\mathfrak{A}}$ is a function from $B_1 \times \cdots \times B_k$ into $\{T, F\}$ such that $C_{\mathfrak{A}}(x_1, \ldots, x_k) = T$ iff $\langle x_1, \ldots, x_k\rangle \in f_c(\mathfrak{A})$.

DEFINITION 8 Let A be a set. If $x \in A$, then the *mark* of x is 0. If $x \subseteq A^n$, $n > 0$, the *mark* of x is n.

DEFINITION 9 Let $\mathfrak{A}$ be a model with universe A.

- If C is a k-place logical predicate, then the *type* of $C_{\mathfrak{A}}$ is

$$\underbrace{\langle 0, \ldots, 0\rangle}_{k \text{ times}} = 0^k.$$

- If C is a k-place logical quantifier and $x = \langle x_1, \ldots, x_k\rangle \in \mathrm{Dom}(C_{\mathfrak{A}})$, then the *type* of $C_{\mathfrak{A}}$ is $\langle t_1, \ldots, t_k\rangle$, where for $1 \leq i \leq k$, t_i is the mark of x_i (see definition 8).

I sum up the mark/type classification in table 4.1.

I now state a theorem establishing a one-to-one correspondence between α-operators and logical predicates and quantifiers of UL restricted to an arbitrary model $\mathfrak{A}$ of cardinality α.

THEOREM 1 Let $\mathfrak{A}$ be a model with a universe A of cardinality α. Let $\mathscr{C}|\mathfrak{A}$ be the set of all logical predicates and quantifiers of UL restricted to $\mathfrak{A}$. Let $\mathscr{C}_\alpha$ be the set of all α-operators. Then there exists a 1-1 function h from

Table 4.1
The mark/type classification

Mark	Type
α-individual: 0	k-place α-predicate operator, p_α: 0^k
n-place α-predicate: n	k-place α-quantifier, $q_\alpha : \langle t_1, \ldots, t^k \rangle$*
$x \in A : 0$	k-place logical predicate, $P_{\mathfrak{A}} : 0^k$
$x \subseteq A^n : n$	k-place logical quantifier, $Q_{\mathfrak{A}} : \langle t_1, \ldots, t_k \rangle$†

*Here t_i, $1 \leq i \leq k$, is the mark of $r_i(\alpha)$, where $[R(\alpha) = \langle r_1(\alpha), \ldots, r_k(\alpha)\rangle] \in \mathrm{Dom}(q_\alpha)$.
†Here t_i, $1 \leq i \leq k$, is the mark of x_i, where $\langle x_1, \ldots, x_k \rangle \in \mathrm{Dom}(Q_{\mathfrak{A}})$. (I assume that an empty n-place relation has a different mark from an empty m-place relation, where $n \neq m$.)

$\mathcal{O}_\alpha$ onto $\mathscr{C}|\mathfrak{A}$ defined as follows: For every $o_\alpha \in \mathcal{O}_\alpha$, $h(o_\alpha)$ is the logical term $C_{\mathfrak{A}}$ such that

- o_α and $C_{\mathfrak{A}}$ are of the same type;
- if $\langle s_1, \ldots, s_k \rangle$ is a k-tuple in $\mathrm{Dom}(C_{\mathfrak{A}})$, then $C_{\mathfrak{A}}(s_1, \ldots, s_k) = o_\alpha[\langle i(s_1), \ldots, i(s_k)\rangle]$, where for some indexing I of A by α, $i(s_1), \ldots, i(s_k)$ are the index images of $s_1, \ldots, s_k$, respectively, under I.

Proof See the appendix.

I symbolize the α-operator correlated with $C_{\mathfrak{A}}$ as o_α^C.

Let me give a few examples of the α-operators corresponding to logical predicates and quantifiers restricted to an arbitrary model $\mathfrak{A}$ with a universe A of cardinality α. I will define the α-counterparts of the logical predicates and quantifiers of the examples in chapter 3.

(37) The identity relation $I_{\mathfrak{A}}$ corresponds to o_α^I, an α-predicate of type $\langle 0, 0 \rangle$, defined by $o_\alpha^I(X) = \mathrm{T}$ iff for some $\beta \in \alpha$, $X = [\langle \beta, \beta \rangle]$.

(38) The universal quantifier $\forall_{\mathfrak{A}}$ corresponds to $o_\alpha^\forall$, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^\forall(X) = \mathrm{T}$ iff $X = [\alpha]$.

(39) The existential quantifier $\exists_{\mathfrak{A}}$ corresponds to $o_\alpha^\exists$, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^\exists(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $s \neq \varnothing$, $X = [s]$.

(40) The cardinal quantifiers $C_{\mathfrak{A}}$ correspond to o_α^δ, α-quantifiers of type $\langle 1 \rangle$, defined by $o_\alpha^\delta(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|s| = \delta$, $X = [s]$.

(41) The quantifiers "finitely many" and "uncountably many," $\mathrm{FIN}_{\mathfrak{A}}$ and $\mathrm{UNC}_{\mathfrak{A}}$, correspond to o_α^{FIN} and o_α^{UNC}, α-quantifiers of type $\langle 1 \rangle$, defined by $o_\alpha^{\mathrm{FIN}}(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|s| < \aleph_0$, $X = [s]$; $o_\alpha^{\mathrm{UNC}}(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|x| > \aleph_0$, $X = [s]$.

(42) The quantifier "as many as not," $\mathrm{MN}_{\mathfrak{A}}$, corresponds to o_α^{MN}, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^{\mathrm{MN}}(X) = T$ iff for some $s \subseteq \alpha$ such that $|s| \geq |\alpha - s|$, $X = [s]$.

(43) The 1-place quantifier "most," $\mathrm{M}^1_{\mathfrak{A}}$, corresponds to $o_\alpha^{\mathrm{M}^1}$, an α-quantifier of type $\langle 1 \rangle$, defined by $o_\alpha^{\mathrm{M}^1}(X) = \mathrm{T}$ iff for some $s \subseteq \alpha$ such that $|s| > |\alpha - s|$, $X = [s]$.

(44) The 2-place quantifier "most," $\mathrm{M}^{1,1}_{\mathfrak{A}}$, corresponds to $o_\alpha^{\mathrm{M}^{1,1}}$, an α-quantifier of type $\langle 1, 1 \rangle$, defined by $o_\alpha^{\mathrm{M}^{1,1}}(X) = \mathrm{T}$ iff for some $s, t \subseteq \alpha$ such that $|s \cap t| > |s - t|$, $X = [\langle s, t \rangle]$.

(45) The 1-place "well-ordering" quantifier $\mathrm{WO}_{\mathfrak{A}}$ corresponds to o_α^{WO}, an α-quantifier of type $\langle 2 \rangle$, defined by $o_\alpha^{\mathrm{WO}}(X) = \mathrm{T}$ iff for some $r \subseteq \alpha^2$ such that r well-orders $\mathrm{Fld}(R)$, $X = [r]$.

(46) The (second-level) set-membership quantifier $\mathrm{SM}_{\mathfrak{A}}$ corresponds to o_α^{SM}, an α-quantifier of type $\langle 0, 1 \rangle$, defined by $o_\alpha^{\mathrm{SM}}(X) = \mathrm{T}$ iff for some $\beta \in \alpha$ and $s \subseteq \alpha$ such that $\beta \in s$, $X = [\langle \beta, s \rangle]$.

(47) The quantifier "ordering of the natural numbers with zero," $\mathrm{NZ}_{\mathfrak{A}}$, corresponds to o_α^{NZ}, an α-quantifier of type $\langle 2, 0 \rangle$, defined by $o_\alpha^{\mathrm{NZ}}(X) = \mathrm{T}$ iff for some $r \subseteq \alpha^2$ and $\beta \in \alpha$ such that $\langle \mathrm{Fld}(r), r, \beta \rangle \cong \langle \omega, <, 0 \rangle$, $X = [\langle r, \beta \rangle]$.

(48) The "the" quantifier, $\mathrm{THE}_{\mathfrak{A}}$, corresponds to o_α^{THE}, an α-quantifier of type $\langle 1, 1 \rangle$, defined by $o_\alpha^{\mathrm{THE}}(X) = \mathrm{T}$ iff for some $s, t \subseteq \alpha$ such that $|s| = 1$ and $s \subseteq t$, $X = [\langle s, t \rangle]$.

I define logical operators as follows:

DEFINITION 10 A *logical operator* of type t is a function that assigns to each cardinal α an α-operator of type t.

3 Unrestricted First-Order Logic: Syntax and Semantics

I can now delineate the syntax and the semantics of first-order logic with Tarskian logical terms satisfying the metatheoretical requirements specified in chapter 3 and defined by means of *logical operators*. As before, I will leave logical functors and quantifier functors out for the sake of simplicity.

Syntax

Let me first present the preliminary notion of the *type of a constant*. A type t is, recall, a sequence of natural numbers $\langle t_1, \ldots, t_k \rangle$, where k is a positive integer. Intuitively, the type of a constant gives us information about its arguments.

- Individual constants do not have a type (since they do not have arguments).
- The type of logical and nonlogical k-place first-level predicates is $\underbrace{\langle 0, \ldots, 0 \rangle}_{k \text{ times}} = 0^k$.
- The type of k-place quantifiers is $\langle t_1, \ldots, t_k \rangle$, where for some $1 \leq i \leq n$, $k_i > 0$. (Intuitively, if the ith argument of a k-place quantifier Q is a singular term, $t_i = 0$; if the ith argument is an n-place first-level predicate, $t_i = n$.)

Primitive symbols

1. Logical symbols
   a. sentential connectives: any collection that semantically forms a complete system of truth-functional connectives, say $\sim$, &, $\vee$, $\rightarrow$, $\leftrightarrow$
   b. n logical predicates and/or quantifiers, $C_1, \ldots, C_n$ of types $t_1, \ldots, t_n$ respectively, $n > 0$
2. Variables: $x_1, x_2, \ldots$ (informally: x, y, z, v, w)
3. Punctuation symbols: (a) parentheses: (,); (b) comma: ,.
4. Nonlogical symbols
   a. individual constants: $a_1, \ldots, a_m$, $m \geq 0$
   b. predicate constants: for each $n > 0$, a finite (possibly empty) set of n-place predicates, $P_1^n, \ldots, P_m^n$.

Well-formed formulas (wffs)

1. Terms: Individual constants and variables are terms.
2. Atomic wffs: If S is an n-place predicate (logical or nonlogical) and $s_1, \ldots, s_n$ are terms, then $S(s_1, \ldots, s_n)$ is an atomic wff.
3. Wffs
   a. An atomic wff is a wff.
   b. If Φ, Ψ are wffs, then so are $(\sim\Phi)$, $(\Phi \,\&\, \Psi)$, $(\Phi \vee \Psi)$, $(\Phi \rightarrow \Psi)$ and $(\Phi \leftrightarrow \Psi)$.
   c. If Q is a quantifier of type $t = \langle t_1, \ldots, t_k \rangle$, n is the maximum of $\{t_1, \ldots, t_k\}$, $x_1, \ldots, x_n$ are distinct variables, and $B_1, \ldots, B_k$ are expressions such that for each $1 \leq i \leq k$, if $t_i = 0$, B_i is a term and otherwise B_i is a wff, then $((Qx_1, \ldots, x_n)(B_1, \ldots, B_k))$ is a wff.

I follow the convention that outermost parentheses in wffs may be omitted.

Bound and free occurrences of variables in wffs I say that x occurs in an expression e iff either $x = e$ or x is a member of the sequence of primitive symbols constituting e.

- There are no bound occurrences of variables in terms.
- If Φ is an atomic wff, then no occurrence of x in Φ is bound.
- If Φ is a wff of the form $\sim\Psi$, then an occurrence of x in Φ is bound iff it is bound in Ψ.
- If Φ is a wff of the form $\Psi \,\&\, \Xi$, $\Psi \vee \Xi$, $\Psi \rightarrow \Xi$, or $\Psi \leftrightarrow \Xi$, then an occurrence of x in Φ is bound iff it is either a bound occurrence in Ψ or it is a bound occurrence in Ξ.
- If Φ is a wff of the form $(Qx_1, \ldots, x_n)(B_1, \ldots, B_k)$, where Q is of type $\langle t_1, \ldots, t_k \rangle$, then an occurrence of x in Φ is bound iff it is an occurrence in some B_i, $1 \leq i \leq k$, such that either x is bound in B_i or for some $1 \leq m \leq t_i$, $x = x_m$.
- An occurrence of x in Φ is free iff it is not bound.

The idea is that if Q is, say, of type $\langle 1, 2, 0 \rangle$ and R_1, R_2 are two 2-place predicates of the language, then in the wff

$$(Qx, y)(R_1(x, y), R_2(x, y), x)$$

Q binds the first two occurrences of x and the second occurrence of y, but the third occurrence of x and the first occurrence of y are free. To make the notation more transparent, I sometimes indicate the type of a quantifier Q with a superscript. That involves rewriting the formula above, for example, as

$$(Q^{1,2,0}x, y)(R_1(x, y), R_2(x, y), x).$$

Sentences A *sentence* is a wff in which no variable occurs free.

In practice I will sometimes omit commas separating the variables in a quantifier expression. Thus instead of (Qx, y), I will write (Qxy). I will also use various types of parentheses.

Semantics

Let $\mathscr{L}$ be a first-order logic with syntax as defined above. Say $\mathscr{L}$ has logical predicates $L_1, \ldots, L_l$, logical quantifiers $Q_1, \ldots, Q_q$, nonlogical [illegible]

P_{n_4}. Each logical predicate or quantifier C of type t is semantically defined by means of a logical operator o^C of the same type.

Let $\mathfrak{A}$ be a model for the language with universe A of cardinality α, defined relative to the nonlogical vocabulary of $\mathscr{L}$ in the usual way. That is, $\mathfrak{A} = \langle A, a_1^{\mathfrak{A}}, \ldots, a_{n_3}^{\mathfrak{A}}, P_1^{\mathfrak{A}}, \ldots, P_{n_4}^{\mathfrak{A}} \rangle$. Let g be an assignment of elements in A to the variables of the language. I define an extension of g, $\bar{g}$, to the terms of the language as follows: For a variable x, $\bar{g}(x) = g(x)$. For an individual constant a, $\bar{g}(a) = a^{\mathfrak{A}}$.

DEFINITION OF SATISFACTION $\mathfrak{A}$ satisfies the wff Φ with the assignment g—$\mathfrak{A} \models \Phi[g]$—iff the following conditions hold:

1. Atomic wffs
   a. Let P be an n-place nonlogical predicate and $s_1, \ldots, s_n$ terms. Then

      $\mathfrak{A} \models P(s_1, \ldots, s_n)[g]$ iff $\langle \bar{g}(s_1), \ldots, \bar{g}(s_n) \rangle \in P^{\mathfrak{A}}$.

      (As before, I identify a 1-tuple with its member.)
   b. Let V be an n-place logical predicate and $s_1, \ldots, s_n$ terms. Then

      $\mathfrak{A} \models V(s_1, \ldots, s_n)[g]$ iff there is an indexing I of A by α such that $o_\alpha^V[\langle i(\bar{g}(s_1)), \ldots, i(\bar{g}(s_n)) \rangle] = \mathrm{T}$,

      where for $1 \leq j \leq n$, $i(\bar{g}(s_j))$ is the index image of $\bar{g}(s_j)$ under I. (See definition 1.)
2. Nonatomic wffs
   a. Let Φ, Ψ be wffs.

      $\mathfrak{A} \models \sim\Phi[g]$ iff $\mathfrak{A} \not\models \Phi[g]$;
      $\mathfrak{A} \models (\Phi \mathbin{\&} \Psi)[g]$ iff $\mathfrak{A} \models \Phi[g]$ and $\mathfrak{A} \models \Psi[g]$
      ...
   b. Let Q be a quantifier of type $\langle t_1, \ldots, t_k \rangle$, let n be the maximum of $\{t_1, \ldots, t_k\}$, let $x_1, \ldots, x_n$ be distinct variables, and let $B_1, \ldots, B_k$ be expressions such that for each $1 \leq j \leq k$, if $t_j = 0$, B_j is a term, and otherwise B_j is a wff. Then

      $\mathfrak{A} \models (\mathrm{Q}x_1, \ldots, x_n)(B_1, \ldots, B_k)[g]$ iff there is an indexing I of A by α such that $o_\alpha^{\mathrm{Q}}[\langle i(\bar{g}_{\bar{x}_{t_1}}(B_1)), \ldots, i(\bar{g}_{\bar{x}_{t_k}}(B_k)) \rangle] = \mathrm{T}$,

      where for $1 \leq j \leq k$,

      if $t_j = 0$, then $\bar{g}_{\bar{x}_{t_j}}(B_j) = \bar{g}(B_j)$;
      if $t_j \geq 1$, $g_{\bar{x}_{t_j}}(B_j) = \{\langle a_1, \ldots, a_{t_j} \rangle \in A^{t_j} : \mathfrak{A} \models$
      $B_j[g(x_1 \ldots$ [illegible]

DEFINITION OF TRUTH IN A MODEL Let $\mathscr{L}$ and $\mathfrak{A}$ be as above. Let Φ be a sentence of $\mathscr{L}$. Then Φ is *true* in $\mathfrak{A}$—$\mathfrak{A} \models \Phi$—iff for some assignment g of elements in A to the variables of the language, $\mathfrak{A} \models \Phi[g]$.

Examples Let $\mathfrak{A}$ be a model with the Group as its universe. Let P and M be the 1-place predicates "x is a philosopher" and "x is a mathematician" respectively, $P^{\mathfrak{A}} = \{\text{Alan}\}$, and $M^{\mathfrak{A}} = \{\text{Alan, Jane}\}$. Let G be the 2-place relation "x graduated from Columbia College before y" and $G^{\mathfrak{A}} = \{\langle \text{Ian, Carl} \rangle, \langle \text{Carl, Gary} \rangle, \langle \text{Ian, Gary} \rangle\}$. The quantifiers !1 ("there is exactly one"), $\mathrm{M}^{1,1}$("most ___ are...."), and TL-F ("three individuals stand in the linear relation ___, the first being..."), restricted to $\mathfrak{A}$, are definable by the following 10-operators:

(49) $o_{10}^{!1} : \mathrm{EQ}(\boldsymbol{P}(10)) \to \{\mathrm{T, F}\}$, where $o_{10}^{!1}[X] = \mathrm{T}$ iff X is similar to $\{0\}$.

(50) $o_{10}^{\mathrm{M}^{1,1}} : \mathrm{EQ}(\boldsymbol{P}(10) \times \boldsymbol{P}(10)) \to \{\mathrm{T, F}\}$, where $o_{10}^{\mathrm{M}^{1,1}}[\langle X, Y \rangle] = \mathrm{T}$ iff $|X \cap Y| > |X - Y|$.

(51) $o_{10}^{\mathrm{TL-F}} \mathrm{EQ}(\boldsymbol{P}(10^2) \times 10) \to \{\mathrm{T, F}\}$, where $o_{10}^{\mathrm{TL-F}}[\langle R, x \rangle] = \mathrm{T}$ iff $\langle R, x \rangle$ is similar to $\langle \{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 0, 2 \rangle\}, 0 \rangle$.

EQ(Z) is the set of all equivalence classes of members of Z under the relation of similarity.

Let I be any indexing of the Group by 10, say indexing by alphabetical order of members' first names. Then

(52) There is exactly one philosopher,

or formally,

(53) $(!1x)Px$,

is true in $\mathfrak{A}$, since $i(P^{\mathfrak{A}}) = \{0\}$.

(54) There is exactly one mathematician,

or,

(55) $(!1x)Mx$,

is false in $\mathfrak{A}$, since $i(M^{\mathfrak{A}}) = \{0, 9\}$ is not similar to $\{0\}$.

(56) Most philosophers are also mathematicians.

or,

(59) $(\text{TL-}Fxy)(Gxy, \text{Ian})$,

is true in $\mathfrak{A}$, because $\langle\{\langle 8, 2\rangle, \langle 2, 6\rangle, \langle 8, 6\rangle\}, 8\rangle$ is similar to $\langle\{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 0, 2\rangle\}, 0\rangle$.

4 Higginbotham and May's Relational Quantifiers

My characterization of logical terms as logical operators puts all logical predicates and quantifiers on a par. It captures a basic principle of logicality, namely that to be logical is to take only structure into consideration. Also captured is the complementary principle that every structure is mirrored by some logical term. It is, however, interesting to divide the expanse of logical terms into groups according to significant characteristics. Mostowski's work allows us to single out predicative quantiers by identifying a method of individuation particular to these quantifiers. In "Questions, Quantifiers, and Crossing" (1981) J. Higginbotham and R. May distinguish four groups of relational quantifiers of the simplest kind, type $\langle 2\rangle$, by means of the *invariance conditions* they satisfy. Their criterion orders simple relational quantifiers according to their complexity, from quantifiers that can only distinguish the number of pairs a binary relation R contains to "fine-grained" quantifiers that take into account the inner structure of R.

Given a universe A, Higginbotham and May define a binary relational quantifier over A as a function $q : P(A \times A) \rightarrow \{T, F\}$. They consider the following invariance conditions:[1]

a. Invariance under automorphisms of A × A
b. (1) Invariance under 1-automorphisms of A × A
   (2) Invariance under 2-automorphisms of A × A
c. Invariance under pair-automorphisms of A × A
d. Invariance under automorphisms of A

Given a set A, $m : A \times A \rightarrow A \times A$ is a (set) *automorphism* of $A \times A$ iff m is a permutation of $A \times A$.

An automorphism $m : A \times A \rightarrow A \times A$ is a 1-*automorphism* of $A \times A$ iff for all $a, b, a', b', c, d, c', d' \in A$,

$$m(a, b) = (a', b') \text{ and } m(c, d) = (c', d') \text{ implies } (a = c \text{ iff } a' = c').$$

That is, m is a 1-automorphism of $A \times A$ iff there is an automorphism m_1 of A such that for all $a, b \in A$,

$$m(a, b) = (m_1(a), b')$$

for some $b' \in A$. Informally, if p_1 and p_2 are pairs with the same first element, then a 1-automorphism m will assign to p_1 and p_2 pairs that also share their first element. In such a case I will say that m respects first elements.

An automorphism $m : A \times A \rightarrow A \times A$ is a 2-*automorphism* of $A \times A$ iff for all $a, b, a', b', c, d, c', d' \in A$,

$$m(a, b) = (a', b') \text{ and } m(c, d) = (c', d') \text{ implies } (b = d \text{ iff } b' = d').$$

That is, m is a 2-automorphism of $A \times A$ iff there is an automorphism m_2 of A such that for all $a, b \in A$,

$$m(a, b) = (a', m_2(b))$$

for some $a' \in A$. Informally, m respects second elements.

An automorphism $m : A \times A \rightarrow A \times A$ is a *pair-automorphism* of $A \times A$ iff m is both a 1-automorphism of $A \times A$ and a 2-automorphism of $A \times A$. That is, m is a pair-automorphism of $A \times A$ iff there are automorphisms m_1, m_2 of A such that for all $a, b \in A$,

$$m(a, b) = (m_1(a), m_2(b)).$$

In such a case I will say that m respects both first and second elements.

The invariance conditions (a) to (d) increasingly extend the notion of relational quantifier, with (a) reflecting a minimalist approach and (d) a maximalist approach. All quantifiers satisfying (a), (b), or (c) satisfy (d), but some quantifiers satisfying (d) do not satisfy (a) to (c); some quantifiers satisfying (c) do not satisfy either (b.1) or (b.2), etc. The more invariance conditions a quantifier satisfies, the less distinctive it is. A quantifier satisfying (a), for instance, does not distinguish between relations that have the same number of elements but otherwise differ in structure (for example, the one is a well-ordering relation, while the other is not). Quantifiers satisfying (d) are those for which I developed my "constructive" definition. Ipso facto, all quantifiers satisfying Higginbotham and May's conditions fall under my definition. Let us describe the quantifiers in each of Higginbotham and May's categories.

Invariance condition (a) The relational quantifiers satisfying (a) constitute an immediate extension of Mostowski's quantifiers and are definable by his cardinality functions. These quantifiers treat relations as sets, and elements of relations, i.e., n-tuples of individuals, as individuals. I will call these *weak relational quantifiers*.

The contribution of weak relational quantifiers to the expressive power of first-order logic is straightforward. They allow us to enumerate the elements of relations: "____ pair(s) of individuals in the universe stand(s) in the binary relation R," and similarly for n-place relations. Thus we can define the 1-place weak relational quantifier

(60) $(\text{Most}^1\ xy)Rxy$

("Most pairs of individuals in the universe fall under the relation R") by the same function t that defines the 1-place predicative "most." Similarly, the 2-place relational "most,"

(61) $(\text{Most}^{1,1}\ xy)(Rxy, Sxy)$

("Most pairs standing in the relation R stand in the relation S"), is defined by the same cardinality function as the 2-place predicative "most."

Weak relational quantifiers do not exhaust the cardinality properties of relations, however. Among the cardinality properties not expressible by weak relational quantifiers is the following:

(62) The (binary) relation R has α elements in its domain,

where α is a cardinal number. Instances of (62) can be stated using a pair of predicative quantifiers:

(63) $(!\alpha\ x)(\exists y)Rxy$

But no weak relational quantifier is equivalent to the pair $(!\alpha\ x)(\exists y)$.

Invariance condition (b) The relational quantifiers satisfying invariance condition (b) essentially say how many individuals in the universe stand to how many individuals in a given relation R. The difference between the two conditions (b.1) and (b.2) is in the direction from which the relation is perceived. Quantifiers satisfying the first condition basically say that α objects in the universe are such that each stands in the relation R to β objects in the universe. Quantifiers satisfying the second condition say that there are β objects in the universe to each of which α objects in the universe stand in the relation R. (The properties predicated on relations by quantifiers satisfying (b.1) and (b.2) can be more complex than those described above, but for my purposes it suffices to consider the basic properties.) Since the two conditions under (b) are symmetrical, it is enough to discuss just one. Following Higginbotham and May, I will concentrate on the first. Higginbotham and May prove that all quantifiers satisfying (b) assign cardinality properties to relations in their scope. A detailed description and proof of their claim appears in the appendix.

Intuitively, we arrive at the cardinality counterparts of quantifiers satisfying invariance condition (b.1) in the following way: Given a model $\mathfrak{A}$ with a universe A of cardinality α and a binary relation $R \subseteq A^2$, we can describe R from the point of view of its cardinalities by stating, with respect to each element of A, to how many objects in A it stands in the relation R and to how many objects in A it does not stand in the relation R. We can thus represent the cardinalities of R by means of a function

$f: \alpha \rightarrow (\beta, \gamma)_\alpha$,

where α serves as a set of indices for the elements of A (as in section 2 above) and $(\beta, \gamma)_\alpha$ is the set of all pairs of cardinals β, γ whose sum is α. Given an element $a_\delta \in A$, $f(\delta)$ is the pair of cardinals $\langle \beta, \gamma \rangle$ such that a_δ stands in the relation R to β individuals and a_δ does not stand in the relation R to γ individuals. But quantifiers do not distinguish which elements of A are associated with a given pair of cardinals $\langle \beta, \gamma \rangle$. Therefore, Higginbotham and May construct equivalence classes of functions f under a similarity relation. Quantifiers are then defined as functions from such equivalence classes to truth values. As you can see, there is a certain resemblance between Higginbotham and May's cardinality functions and my α-operators. Indeed, I arrived at the idea of my definition by generalizing Higginbotham and May's method.

Invariance condition (c) Quantifiers invariant under pair-automorphisms of $A \times A$ distinguish identities and nonidentities both in the domain and in the range of a given relation R. These quantifiers can express such properties of relations having to do with identities as, e.g., "____ is a one-to-one relation."

Invariance condition (d) I will call relational quantifiers satisfying invariance under automorphisms of A, but not the other invariance conditions, *strong relational quantifiers*. Strong relational quantifiers are genuine logical terms, and they can be represented by logical operators defined in section 2 above. These quantifiers make the finest distinctions among relations that logical terms are capable of making. Below I will give several examples of strong relational quantifiers in natural language, and also of weaker relational quantifiers satisfying (a) through (c).

5 Linguistic Applications

Several "types" of logical terms of UL have received ample attention in logico-linguistic circles, usually under the heading of "generalized quantifiers." In chapter 2 we saw Mostowskian quantifiers being used to interpret determiners. In the present section I will further expand the domain of applications of UL quantifiers. My discussion will not assume the form of a survey. Instead, I will describe applications of logical quantifiers that came up in the course of my own investigations. (Other works devoted to linguistic applications of, or theoretical linguistic approaches to, generalized quantifiers are listed in the references. The reader is referred to Barwise and Cooper, Higginbotham and May, Keenan, Keenan and Moss, Keenan and Stavi, May, van Benthem, and Westerståhl, among others.)

I will begin with a new application of Mostowskian quantifiers and then proceed through Higginbotham and May's categories to describe increasingly strong relational quantifiers in natural language.

Generalized operations on relations

In standard first-order logic we use the existential and universal quantifiers as operators that, given two binary relations R and S, yield new relations called the relative product of R and S— $R|^{P}S$ —and the relative sum of R and S—$R|^{S}S$. These are defined (by dual conditions) as follows:

$$R|^{P}S =_{df} \{\langle x, y\rangle : (\exists z)(xRz \,\&\, zSy)\}$$

$$R|^{S}S =_{df} \{\langle x, y\rangle : (\forall z)(xRz \vee zSy)\}$$

Linguistically, we can interpret the relation "being a paternal uncle of" as the relative product of the relations "being a brother of" and "being a father of," etc. By generalizing the definitions of relative product and sum, we arrive at the notion of a relative product/sum modulo Q, where Q is a 1-place Mostowskian quantifier. I define the relative product and sum of binary relations R and S modulo Q as follows:

$$R|^{P}_{Q}S =_{df} \{\langle x, y\rangle : (Qz)(xRz \,\&\, zSy)\}$$

$$R|^{S}_{Q}S =_{df} \{\langle x, y\rangle : (Qz)(xRz \vee zSy)\}$$

(As in the traditional product and sum, if Q_1 is the dual of Q_2, the definiens of $R|^{P}_{Q_1}S$ is the dual of the definiens of $R|^{S}_{Q_2}S$.) I will call the standard relative product the relative product modulo ∃ and the standard relative sum the relative sum modulo ∀. The notions of relative product and sum allow us to define relations that include a "cardinality factor." The operation of relative product modulo Q appears to be especially useful, as can be seen in the following examples:

(64) x is a friend of many people who know y.

(65) x has few common acquaintances with y.

When R is an ordering relation, we can define relations that have to do with distance or relative position in R as relative products of R modulo

the appropriate Q. In this way we can define

(66) There are n elements between x and y in R.

(67) x is far behind/ahead-of y in R.

(68) x is second best to y in P.

Here P is a property (e.g., diving) that determines the field of an implicit ordering relation R, "being better at . . . ."

Two-place predicative quantifiers can also be used to define sets and relations that include a cardinality factor. I call the operation of constructing such a set (or relation) from two initial relations R and R' "a generalized relative product of R and R'." For example, using the quantifier "same number," defined in the obvious way, we can single out the median element in a linear ordering relation with

(69) (same-number z)(xRz, zRx).

In a similar way, we can define "x is relatively high/low in R."

It is often useful to consider "semilinear" orderings, an ordering like a linear ordering but with the requirement "$(\forall x)(\forall y)(x < y \vee y > x \vee x = y)$" replaced by "$(\forall x)(\forall y)(x < y \vee y > x \vee x \approx y)$," where $\approx$ is some equivalence relation, for example "being in the same income bracket as." Thus if R is a semilinear ordering relative to "being in the same income bracket as," (69) will give us the set of all elements in the middle income bracket. Using a second predicative quantifier, we can now express statements indicating how many individuals occupy a certain relative position in R. For example,

(70) Proportionally more women hold high-paying jobs in San Diego than in other cities in the country.

Other statements stating formal properties of generalized relative products of R and S can be constructed using relational quantifiers defined in this chapter.

Weak relational quantifiers

I will indicate some of the uses of weak relational quantifiers. Given a relative product modulo Q, e.g., (66), we can use weak relational quantifiers to make statements of the form

(71) There are m pairs whose distance in R is n.

Other cases of quantification where pairs are taken as basic units are also naturally expressed using weak relational quantifiers. For example,

(72) Most divorced couples do not remarry.

Consider, however,

(73) Four married couples left the party.

The most natural construal of (73) as a weak relational quantification fails. Suppose that "exactly 4," !4, is a 2-place weak relational quantifier over binary relations. Then, since !4 is essentially a Mostowskian quantifier, we can define it by a cardinality function as described in chapter 2. That is, given a universe A, $t_A^{!4}$ is a function such that for any quadruple $\alpha, \beta, \gamma, \delta$, where $\alpha + \beta + \gamma + \delta = |A|$,

$t_A^{!4}(\alpha, \beta, \gamma, \delta) = T$ iff $\alpha = 4$.

This means that if R and S are binary relations on A,

$!4(R, S) = T$ iff $|R \cap S| = 4$.

Now, if we interpret (73) as

(74) $(!4\ xy)(x$ is married to y, x and y left the party),

then (74) turns out true when the number of married couples who left the party is two, not four. (This is because there are two pairs in a couple.) Thus (74) is an incorrect rendering of (73). There are various remedies to the problem. Among them are the following:

a. We can treat binary relations as *sets of couples* (a couple being an unordered pair) and then define weak relational quantifiers as regular Mostowskian quantifiers by setting numerical conditions on the atoms of the Boolean algebra generated by n-tuples of such "sets" in a given universe A. The couple quantifier !4 will thus be defined by the same t-function as the corresponding quantifier based on pairs: $!4(R, S) = T$ iff the intersection of the two sets of couples R and S yields a set of 4 couples.

b. We can construe couple quantifiers as strong relational quantifiers, i.e., quantifiers satisfying invariance condition (d).

By adopting strategy (a), we will be able to use weak relational quantifiers to symbolize the following English sentences:

(75) Half the students in my class do not know each other.

(76) Most of my friends have few common acquaintances.

(77) Few townsmen and villagers hate each other.

(78) Almost all brothers compete with each other.

Thus, for instance, (75) will be symbolized as

(79) (Half xy)[x is a student in my class & y is a student in my class & $x \neq y$, $\sim$(x knows y & y knows x)].

But to interpret

(80) Most younger brothers envy their elder brothers

we must go back to quantifiers based on pairs.

I should say that weak relational quantifiers (based on pairs or on couples) do not exhaust the possibilities of interpretation of the sentences in our examples. On my interpretation, (75), for instance, is true if my class consists of four students, a, b, c, and d, and one of the students, say a, does not know (and is not known to) anyone in the class, but the rest—b, c, and d—all know each other. Someone may wish to interpret (75) so that it will come out false in the situation just described. This can be done by adopting stronger relational quantifiers.

Linearity quantifiers

Higginbotham and May's 1-place relational cardinality quantifiers over a universe A, i.e., 1-place binary quantifiers invariant under 1-automorphism of $A \times A$, essentially say how many individuals stand to how many individuals in a given binary relation R. But this is exactly what a linear quantifier prefix with two 1-place predicative quantifiers says about a relation R in a model $\mathfrak{A}$ with a universe A. For that reason I name relational cardinality quantifiers *linearity quantifiers*. Higginbotham and May called the operation of constructing a relational quantifier equivalent to a linear quantifier prefix with two predicative quantifiers *absorption*. A relational quantifier constructed by absorption is said to be *separable*.

The rule of absorption is this: if Q_1 and Q_2 are two 1-place predicative quantifiers over a universe A and R is a binary relation included in A^2, then the quantifier prefix $(Q_1 x)(Q_2 y)$ will be absorbed by $(Q_3 xy)$, where Q_3 is a linearity quantifier over A such that

$Q_3(R) = T$ iff $Q_1(\{a \in A : Q_2(\{b \in A : aRb\}) = T\}) = T$.

We can generalize the operation of absorption to n-place quantifier prefixes by defining 1-place linearity quantifiers on n-place relations over a universe A. A 1-place linearity quantifier on an n-place relation over a universe A is a function

$q : P(A^n) \to \{T, F\}$

that is invariant under *linear automorphisms* of A^n. I define "linear automorphism of A^n" as follows. The function

$m : A^n \to A^n$

is a linear automorphism of A^n iff m is an automorphism of A^n and for any $a_1, a_2, \ldots, a_n, a'_1, a'_2, \ldots, a'_n, b_1, b_2, \ldots, b_n, b'_1, b'_2, \ldots, b'_n \in A$ the following holds:

If $m(a_1, a_2, \ldots, a_n) = (a'_1, a'_2, \ldots, a'_n)$ and $m(b_1, b_2, \ldots, b_n) = (b'_1, b'_2, \ldots, b'_n)$, then

1. $a_1 = b_1$ iff $a'_1 = b'_1$, and
2. if $a_1 = b_1$, then $a_2 = b_2$ iff $a'_2 = b'_2$, and

⋮

$n - 1$. if $a_1 = b_1$ and $a_2 = b_2$ and ... and $a_{n-2} = b_{n-2}$, then $a_{n-1} = b_{n-1}$ iff $a'_{n-1} = b'_{n-1}$.

To return to absorption of two linearly ordered 1-place predicative quantifiers, let A be a set of n children, $n \geq 3$. Consider the sentence

(81) Three children had three friends each.

We can formalize (81) with either (82) or (83) below:

(82) $(!3\ x)(!3\ y)$ x is a friend of y.

Here !3 is a 1-place predicative quantifier defined, for A, by a Mostowskian function t such that for any (k, m) in its domain $(k + m = n)$, $t(k, m) = \mathrm{T}$ iff $k = 3$.

(83) $(3/3\ xy)$ x is a friend of y.

Here 3/3 is a linearity quantifier of type $\langle 2 \rangle$ defined, for A, by a Higginbothtam-May function k such that for any $[f]$ in its domain $(f : n \to (j, k)_n)$, $k[f] = \mathrm{T}$ iff f is similar to some f^* such that

$f^*(0) = f^*(1) = f^*(2) = (3, n - 3)$ and

$f^*(3), \ldots, f^*(n - 1) \neq (3, n - 3)$.

Intuitively, the function f^* assigns 3 to children 0, 1, and 2 as the number of their friends, and $n - 3$ as the number of their nonfriends. To all other children f^* assigns a different combination of numbers of friends and nonfriends. (For the sake of simplicity I assumed that a child can have himself or herself as a friend.)

Note, however, that linearity quantifiers on binary relations can also express Boolean combinations, possibly infinite, of linear quantifier prefixes with predicative quantifiers. Thus, consider the following infinite conjunction in which "number" stands for "natural number" and n ranges over the natural numbers:

(84) One number has no predecessors, and two numbers have at most one predecessor, and three numbers have at most two predecessors, and ..., and n numbers have at most $n - 1$ predecessors, and ...

This infinite conjuction cannot be formalized in first-order logic with predicative quantifiers, but it can be formalized in first-order logic with linearity quantifiers on binary relations. I will symbolize it as

(85) (n at most $(n - 1)$ xy)x has a predecessor y,

where "n at most $(n - 1)$" is defined, in a universe A of cardinality $\aleph_0$, by a function

$k : [\mathrm{F}] \to \{\mathrm{T}, \mathrm{F}\}$

such that for any $[f] \in [\mathrm{F}]$, $k[f] = \mathrm{T}$ iff f is similar to the function

$f^* : \aleph_0 \to (k, l)_{\aleph_0}$,

which is such that for every $n < \aleph_0$,

$f^*(n) = (n, \aleph_0)$.

Intuitively, f represents a relation R with field of cardinality $\aleph_0$ such that under some indexing of the universe A by $\aleph_0$, a_0 stands in the relation R to no objects in A, a_1 stands in the relation R to one object in A, a_2 stands in the relation R to two objects in A, and so on. Clearly, k also defines the complex quantifier in (86):

(86) One number has no predecessor, and one number has exactly one predecessor, and one number has exactly two predecessors, and ..., and one number has exactly n predecessors, and ...

Note that k need not express a condition which exhibits a regularity. Using a quantifier k_1 similar (in the intuitive sense) to k, we can represent an irregular situation like the following:

(87) Two children have two friends each, and ten children have four friends each, and twelve children have nine friends each, and ...

Another kind of cardinality condition expressible with linearity quantifiers, but not with a standard prefix of two 1-place predicative quantifiers, is exemplified by the following sentence:

(88) There is a great variance in the number of friends of each of these youngsters

(which could also be phrased as "These youngsters differ considerably in the numbers of their friends"). Assuming, for simplicity, that the universe consists of "these youngsters" and that the friends in question are members of the universe, (88) could be expressed as

(89) (Great variance xy) youngster x has youngster y for a friend,

where for each universe A of cardinality α, "great variance" is defined by the function k such that for every $[f] \in \mathrm{Dom}(k)$,

$k([f]) = \mathrm{T}$ iff there is a wide distribution of cardinals γ

such that for some $\beta \in \alpha, f(\beta) = (\gamma, \alpha - \gamma)$.

We can construct 2-place linearity quantifiers, of type $\langle 1, 2\rangle$, that will enable us to restrict linear quantification to $B \upharpoonright R$ (R with its domain limited to B). If we want to symbolize the following sentence without assuming the universe consists of "these youngsters," we will use the 2-place "great variance" quantifier.

(90) There is a great variance in the number of words in the active vocabulary of each of these youngsters.

This sentence will be rendered "(Great variance xy)(x is one of these youngsters, x has word y in his active vocabulary)."

Let us now turn to absorption of two 2-place predicative quantifiers. A linguistically interesting case is that of quantifications of the form

(91) $(Q_1x)(\Phi, (Q_2y)(\Psi, \Xi))$,

where Φ, Ψ, Ξ are well-formed formulas. The quantifiers in (91) are absorbed by the quantifier $(Q_1/Q_2)^{1,2,2}$, defined, for a universe A, as follows: for every $B \subseteq A$ and $C, D \subseteq A^2$,

$(Q_1/Q_2)_A(B, C, D) = \mathrm{T}$ iff $(Q_1)_A(\{a \in A : a \in B\}$,
$\{a \in A : (Q_2)_A(\{b \in A : \langle a, b\rangle \in C\}, \{b \in A : \langle a, b\rangle \in D\}) = \mathrm{T}\}) = \mathrm{T}$.

It is easy to see that (91) is equivalent to

(92) $((Q_1/Q_2)^{1,2,2}xy)(\Phi, \Psi, \Xi)$,

whose satisfaction condition in a model $\mathfrak{A}$ with a universe A by an assignment g is

$\mathfrak{A} \models (Q_1/Q_2xy)(\Phi, \Psi, \Xi)[g]$ iff $(Q_1)_A(\{a \in A : \mathfrak{A} \models \Phi[g(x/a)]\}$,
$\{a \in A : (Q_2)_A(\{b \in A : \mathfrak{A} \models \Psi[g(x/a)(y/b)]\}$,
$\{b \in A : \mathfrak{A} \models \Xi[g(x/a)(y/b)]\}) = \mathrm{T}\}) = \mathrm{T}$.

This definition of absorption is similar to one proposed by R. Clark and E. L. Keenan in "The Absorption Operator and Universal Grammar" (1986). But there is an essential difference: whereas I constructed the absorption quantifier Q_1/Q_2 in such a way that in the formula

$(Q_1/Q_2)(\Phi x, \Psi xy, \Xi xy)$

Q_1/Q_2 binds all free variables, Clark and Keenan defined Q_1/Q_2 in such a way that it does not bind the occurrence of x in Ψxy. The reason the absorbing quantifier has to bind x in Ψxy is simple: Q_1/Q_2 has to be so defined that

(93) $(Q_1/Q_2xy)(\Phi, \Psi, \Xi)$

is logically equivalent to

(94) $(Q_1x)(\Phi, (Q_2y)(\Psi, \Xi))$,

no matter what well-formed formulas Φ, Ψ, and Ξ are. Now it is an essential feature of (94) that any free occurrence of x in Φ, Ψ, or Ξ is bound by Q_1, and similarly, that any free occurrence of y in Ψ or Ξ is bound by Q_2. The relation of binding between quantifiers and free variables in (94) must be preserved by (93). In particular, if x occurs free in Ψ, it should be bound by Q_1/Q_2. The definition of absorption by Clark and Keenan that I have referred to goes as follows: for every $B, C \subseteq A$ and $D \subseteq A^2$,

$(Q_1/Q_2)_A(B, C, D) = \mathrm{T}$ iff $(Q_1)_A(\{a \in A : a \in B\}$,
$\{b \in A : (Q_2)_A(\{b \in A : b \in C\}, \{b \in A : \langle a, b\rangle \in D\}) = \mathrm{T}\}) = \mathrm{T}$.

This definition is intended to "simulate" quantifications of the form

$(Q_1x)[\Phi x, (Q_2y)(\Psi y, \Xi xy)]$.

But as we have seen, it is not adequate for absorbing all well-formed formulas of the form

$(Q_1x)[\Phi, (Q_2y)(\Psi, \Xi)]$.

Note that the definition of satisfaction allows me to apply my absorbing quantifier whether x occurs free in Ψ or not. For example, I can apply absorption to

(95) Every man loves some woman,

or formally,

(96) $(\forall x)[Mx, (\exists y)(Wy, Lxy)]$,

and get

(97) $(\forall/\exists xy)(Mx, Wy, Lxy)$,

which has the right truth conditions. This is because the truth definition of (97) in a model $\mathfrak{A}$ is

$\mathfrak{A} \models (\forall/\exists xy)(Mx, Wy, Lxy)$ iff $\forall_A[\{a \in A : \mathfrak{A} \models Mx[g(x/a)]\}$,
$\{a \in A : \exists_A(\{b \in A : \mathfrak{A} \models Wy[g(x|a)(y/b)]\}$,
$\{b \in A : \mathfrak{A} \models Lxy[g(x/a), (y/b)]\}) = \mathrm{T}\}] = \mathrm{T}$,

and $\mathfrak{A} \models Wy[g(x/a), (y/b)]\}$ is equivalent to $\mathfrak{A} \models Wy[g(y/b)]\}$.

Absorption operators were originally investigated by Higginbotham and May (1981) in an attempt to account for the logical structure of cross reference, as in the Bach-Peters sentence

(98) Every pilot who shot at it hit some Mig that chased him.

May, in "Interpreting Logical Form" (1989), explains the issue as follows:

> If scope is represented asymmetrically [as it is in formulas of form (91)], then the narrower scope quantifier cannot bind, as a bound variable, the pronoun contained within the broader scope phrase, which, in virtue of having broader scope, is outside its c-command domain. Thus if the *every*-phrase has broader scope, *it* cannot be a variable bound by the narrower *some*-phrase. Of course this problem disappears if the proper structure associated with [(98)] at LF is one of symmetric c-command, since then *it* would reside within the c-command domain of *some Mig that chased him* simultaneously with *him* residing within the c-command domain of *every pilot who shot at it*. [Absorption is then presented as] a structural readjustment of asymmetric structures into symmetric ones.[2]

I will not describe the exchange of views regarding this matter in the linguistic literature.[3] However, I would like to propose for consideration two formalizations of (98) in the spirit of May's suggestion.

First consider the 2-place predicative quantifier $\exists^{*1,1}$, which I will call "the conditional existential quantifier" or "the conditional *some*." Given a universe A, I define $\exists^*_A$ as follows: for any $B, C \subseteq A$,

$\exists^*_A(B, C) = \mathrm{T}$ iff either $B = \varnothing$ or $B \cap C \neq \varnothing$.

In terms of cardinality t-functions (see chapter 2), $\exists^*_A$ is defined by the function $t^{\exists^*}_A$ such that for any $(\alpha, \beta, \gamma, \delta)$ in its domain,

$t^{\exists^*}_A(\alpha, \beta, \gamma, \delta) = \mathrm{T}$ iff either $\beta = 0$ or $\alpha \neq 0$.

Figure 4.1 helps elucidate the relation between $\exists^*$ and $t^{\exists^*}_A$. Clearly, if Φ, Ψ are wffs,

(99) $(\exists^* x)(\Phi, \Psi)$

is logically equivalent to

(100) $(\exists x)\Phi \rightarrow (\exists x)(\Phi \,\&\, \Psi)$.

The quantifier $\exists^*$ might be used to interpret such English sentences as

(101) Every boy who chased a unicorn caught one,

understood as having the same truth conditions as

(102) $(\forall x)\{Bx \rightarrow [(\exists y)(Uy \,\&\, CHxy) \rightarrow (\exists y)(Uy \,\&\, CHxy \,\&\, Cxy)]\}$,

with the obvious symbolization key for B, U, CH and C. The formal

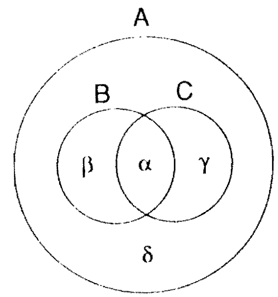

Figure 4.1

sentence (102) is equivalent to

(103) $(\forall x)[Bx \rightarrow (\exists^* y)(Uy \,\&\, CHxy, Cxy)]$,

which in some respects is closer in form to (101). Returning to the Bach-Peters sentence (98), the meaning of (98) seem to be captured by

(104) $(\forall x)\{Px \rightarrow [(\exists y)(My \,\&\, Cyx \,\&\, Sxy) \rightarrow (\exists y)(My \,\&\, Cyx \,\&\, Sxy \,\&\, Hxy)]\}$,

with the obvious readings for P, M, C, S, and H. (In understanding (98) as having the same meaning as (104), I follow Higginbotham and May in "Questions, Quantifiers, and Crossing" and Clark and Keenan in "The Absorption Operator and Universal Grammar."[4]) However, although (104) avoids the problem of cross binding, it does not appear to have the same logical structure as (98). I propose, therefore, that we assign to (98) the logical form

(105) $(\forall x)[Px \rightarrow (\exists^* y)(My \,\&\, Cyx \,\&\, Sxy, Hxy)]$.

Alternatively, we can analyze (98) as

(106) $(\forall x)[Px, (\exists^* y)(My \,\&\, Cyx \,\&\, Sxy, Hxy)]$,

which is obtained from (105) by replacing the 1-place $\forall$ by its 2-place variant. Both (105) and (106) are equivalent to (104), but I think they offer a better semantic representation of (98) than does (104), while solving the problem of cross binding just as well. If absorption is still desirable, we can apply it to the linear pair $\langle \forall, \exists^* \rangle$. We then obtain

(107) $(\forall/\exists^{*1,2,2}xy)(Px, My \,\&\, Cyx \,\&\, Sxy, Hxy)$.

Finally, to increase the structural similarity with (98), we can rewrite (107) using a quantifier equivalent to $\forall/\exists^{*1,2,2}$ but of the type $\langle 1, 2, 2, 2 \rangle$. This quantifier will be so defined that

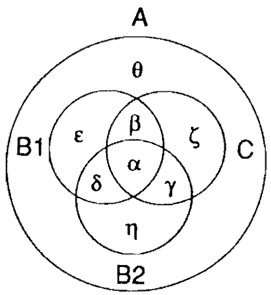

Figure 4.2

(108) $(\forall/\exists^{*1,2,2,2}xy)(Px, Sxy, Hxy, My \,\&\, Cyx)$

is equivalent to (107). Alternatively, we can construct a 3-place variant of $\exists^*$ and replace (106) with

(109) $(\forall x)[Px, (\exists^{*1,1,1}y)(Sxy, Hxy, My \,\&\, Cyx)]$.

The quantifier $\forall/\exists^{*1,2,2,2}$ will then be obtained by absorption from $\langle\forall, \exists^{*1,1,1}\rangle$ in the obvious way. Formally, there is no problem in constructing "superfluous" versions of quantifiers, and indeed, in chapter 2, I noted that such terms are common in natural languages. The 3-place $\exists^*$ is defined by a function t as follows:

$t_A(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta) = \mathrm{T}$ iff either $\delta = 0$ or $\alpha \neq 0$

The relation between $\exists^{*1,1}$ and $\exists^{*1,1,1}$ becomes clear when we compare figure 4.1 to figure 4.2. (Given an x, B1 represents "Sxy," B2 represents "$My \,\&\, Cyx$," and C represents "Hxy.")

If my analysis is correct, it is left for the linguist to account for the occurrence of "superfluous" logical forms in certain natural-language constructions. I will not attempt such an account. It may indeed be the case that what is superfluous from a purely logical point of view is significant from a linguistic viewpoint.

Pair quantifiers

Pair quantifiers are 1-place quantifiers satisfying Higginbotham and May's invariance condition (c) but not (b) or (a). Here are two examples:

(110) Three villagers and two townsmen exchanged blows.

(111) Two Germans and three Americans will challenge each other in the next tournament.

Note that the number words in each of these sentences can themselves be construed as quantifiers. But as predicative quantifiers, neither is within the scope of the other. Therefore, these are not ordinary predicative quantifications but fall under the category of branching quantifications. A general analysis of the branching structure will be given in chapter 5.

Other pair quantifiers express various correspondence relationships. Thus, treating modes of unhappiness as individuals (or allowing ascent to second-order logic), we can analyze Tolstoy's opening to *Anna Karenina* as a pair quantification stating a one-to-one correspondence:

(112) Each unhappy family is unhappy in its own way.

Other examples of pair quantifiers are

(113) Courses vary in the students they attract.

(114) My countrymen are divided in their views about war and peace.

(115) Different students answered different questions on the exam.[5]

Statements of the form "For every A there is a B," discussed by G. Boolos (1981), can also be construed as pair quantifications.

(116) For every drop of rain that falls, a flower grows.[6]

Sentences (112) to (116) include quantifiers that take into account not only cardinalities but more refined formal features of objects standing in relations. In particular, these quantifiers discern sameness and difference between objects within (though not across) each domain of a given relation. Thus the 1-place quantifier "vary," as in

$(\text{Vary } xy)Rxy$,

is defined, for each cardinal α, by a logical operator o_α such that, for example,

$o_{10}^{\text{vary}}([\{\langle 1, 6\rangle, \langle 2, 6\rangle, \langle 3, 6\rangle, \langle 4, 6\rangle, \langle 5, 7\rangle\}]) = \mathrm{F}$,

while

$o_{10}^{\text{vary}}([\{\langle 1, 6\rangle, \langle 2, 7\rangle, \langle 3, 8\rangle, \langle 4, 3\rangle, \langle 5, 9\rangle\}]) = \mathrm{T}$.

Finally, I would like to point out a construction with strong relational quantifiers that is more common in Hebrew than in English. Consider the following situation: A group of objects is divided into pairwise disjoint subgroups of n members each, and a certain condition is set on the members of each group. For example, given an initial group of students, the members of each subgroup are assigned a room in the dormitory, or given an initial group of soldiers (say an army in disarray), the members of each