subgroup "fight their own war." These situations are described concisely in the sentences below.

(117) Every four students will receive a room.

Kol arba'ah studentim yekablu heder.

(118) Every several soldiers fought their own war.

Kol kama hayalim lahamu at milhamtam shelahem.

"Every four" and "every several" (in the sense indicated above) are naturally understood as strong relational quantifiers that distinguish partitions of a certain size in the domain of the quantified relation.

Strong relational quantifiers

Strong relational quantifiers are quantifiers satisfying the strongest invariance condition (d) in Higginbotham and May's list but not (a) through (c). As we have seen above, "couple" quantifiers fall under this category. Other genuinely strong relational quantifiers are quantifiers requiring the detection of sameness and difference across domains. Thus the quantifier "Reflexive xy" is a strong quantifier, as are all quantifiers attributing order properties to relations in their scope. Consider the following examples:

(119) Parenthood is an antireflexive relation.

(120) Forty workers elected a representative from among themselves.

We have completed the description of first-order Unrestricted Logic (UL) based on the philosophical conception developed in chapter 3. This conception was formally and linguistically elaborated in the present chapter. Along with Lindström's original semantics, I have proposed a "constructive" method for representing logical terms with ordinal functions. This method constitutes a natural extension of Mostowski's work on 1-place predicative quantifiers. Some philosophical issues concerning the new conception of logic will be discussed in chapter 6. But first I would like to investigate the impact of the generalization of quantifiers on another new logical theory. This theory has to do not with logical particles but with complex structures of logical particles. It is the theory of *branching quantification.*

Chapter 5
Ways of Branching Quantifiers

1 Introduction

Branching quantifiers were first introduced by L. Henkin in his 1959 paper "Some Remarks on Infinitely Long Formulas." By "branching quantifiers" Henkin meant a new, nonlinearly structured quantifier prefix whose discovery was triggered by the problem of interpreting infinitistic formulas of a certain form.[1] The branching (or partially ordered) quantifier prefix is, however, not essentially infinitistic, and the issues it raises have largely been discussed in the literature in the context of finitistic logic, as they will be here.

We would eventually like to know whether branching quantification is a genuine logical form. But today we find ourselves in an interesting situation where it is not altogether clear what the branching structure is. While Henkin's work purportedly settled the issue in the context of standard quantifiers, Barwise's introduction of new quantifiers into branching theory reopened the question. What happens when you take a collection of quantifiers, order them in an arbitrary partial ordering, and attach the result to a given formula? What truth conditions are to be associated with the resulting expression? Are these conditions compositionally based on the single quantifiers involved? Although important steps toward answering these questions were made by Barwise, Westerståhl, van Benthem, and others, the question is to my mind still open. Following the historical development, I will begin with standard quantifiers.

Initially there were two natural ways to approach branching quantification: as a generalization of the ordering of standard quantifier prefixes and as a generalization of Skolem normal forms.

A generalization of the ordering of standard quantifier prefixes

In standard modern logic, quantifier prefixes are linearly ordered, both syntactically and semantically. The syntactic ordering of a quantifier prefix $\langle(Q_1 x_1), \ldots, (Q_n x_n)\rangle$ (where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq n$) mirrors the sequence of steps used to construct well-formed formulas with that quantifier prefix. Thus, if

(1) $(Q_1 x_1) \ldots (Q_n x_n)\Phi(x_1, \ldots, x_n)$

is a well-formed formula, of any two quantifiers $Q_i x_i$ and $Q_j x_j$ ($1 \leq i \neq j \leq n$), the innermore precedes the outermore in the syntactic construction of (1). The semantic ordering of a quantifier prefix is the order of determining the truth (satisfaction) conditions of formulas with that prefix, and it is the backward image of the syntactic ordering. The truth of a sentence of the form (1) in a model $\mathfrak{A}$ with universe A is determined in the following order of stages:[2]

1. Conditions of truth (in $\mathfrak{A}$) for $(Q_1 x_1)\Psi_1(x_1)$, where
   $\Psi_1 = (Q_2 x_2) \ldots (Q_n x_n)\Phi(x_1, x_2, \ldots, x_n)$
2. Conditions of truth for $(Q_2 x_2)\Psi_2(x_2)$, where
   $\Psi_2 = (Q_3 x_3) \ldots (Q_n x_n)\Phi(a_1, x_2, x_3, \ldots, x_n)$
   and a_1 is an arbitrary element of A

   $\vdots$

n. Conditions of truth for $(Q_n x_n)\Psi_n(x_n)$, where
   $\Psi_n = \Phi(a_1, a_2, \ldots, a_{n-1}, x_n)$
   and $a_1, \ldots, a_{n-1}$ are arbitrary elements of A

We obtain branched quantification by relaxing the requirement that quantifier prefixes be *linearly* ordered and allowing *partial* ordering instead. It is clear what renouncing the requirement of linearity means syntactically. But what does it mean semantically? What would a partially ordered definition of truth for multiply quantified sentences look like? Approaching branching quantifiers as a generalization on the ordering of quantifiers in standard logic leaves the issue of their correct semantic definition an open question.

A generalization of Skolem normal forms

The Skolem normal form theorem says that every first-order formula is logically equivalent to a second-order prenex formula of the form

(2) $(\exists f_1) \ldots (\exists f_m)(\forall x_1) \ldots (\forall x_n)\Phi$,

where $x_1, \ldots, x_n$ are individual variables, $f_1, \ldots, f_m$ are functional variables ($m, n \geq 0$), and Φ is a quantifier-free formula.[3] This second-order formula is a *Skolem normal form*, and the functions satisfying a Skolem normal form are *Skolem functions*.

The idea is roughly that given a formula with an individual existential quantifier in the scope of one or more individual universal quantifiers, we obtain its Skolem normal form by replacing the former with a functional existential quantifier governing the latter. For example,

(3) $(\forall x)(\forall y)(\exists z)\Phi(x, y, z)$

is equivalent to

(4) $(\exists f^2)(\forall x)(\forall y)\Phi[x, y, f^2(x, y)]$.

The functional variable f^2 in (4) replaces the individual variable z bound by the existential quantifier $(\exists z)$ in (3), and the arguments of f^2 are all the individual variables bound by the universal quantifiers governing $(\exists z)$ there. It is characteristic of a Skolem normal form of a first-order formula with more than one existential quantifier that for any two functional variables in it, the set of arguments of one is included in the set of arguments of the other. Consider, for instance, the Skolem normal form of

(5) $(\forall x)(\exists y)(\forall z)(\exists w)\Phi(x, y, z, w)$,

namely,

(6) $(\exists f^1)(\exists g^2)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^2(x, z)]$.

In general, Skolem normal forms of first-order formulas are formulas of the form (2) satisfying the following property:

(7) The functional existential quantifiers $(\exists f_1), \ldots, (\exists f_m)$ can be ordered in such a way that for all $1 \leq i, j \leq m$, if $(\exists f_j)$ syntactically precedes $(\exists f_i)$, then the set of arguments of f_i in Φ is essentially included in the set of arguments of f_j in Φ.[4]

This property reflects what W. J. Walkoe calls the "essential order" of linear quantifier prefixes.[5]

The existence of Skolem normal forms for all first-order formulas is thought to reveal a systematic connection between Skolem functions and existential individual quantifiers. However, this connection is not symmetric. Not all formulas of the form (2), general Skolem forms, are expressible in standard (i.e., linear) first-order logic. General Skolem forms not satisfying (7) are not.

It is natural to generalize the connection between Skolem functions and existential quantifiers into a complete correspondence. But such a generalization requires that first-order quantifier prefixes not be in general linearly ordered. The simplest Skolem form not satisfying (7) is

(8) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^1(z)]$.

Relaxing the requirement of syntactic linearity, we can construct a "first-order" correlate for (8), namely

$$(9)\quad \begin{matrix}(\forall x)(\exists y)\\(\forall z)(\exists w)\end{matrix} \Big> \Phi(x, y, z, w).$$

We see that the semantic structure of a partially ordered quantifier prefix is introduced in this approach *together* with (or even *prior* to) the syntactic structure. The interpretation of a first-order branching formula is fixed to begin with by its postulated equivalence to a second-order, linear Skolem form.

Do the two generalizations above necessarily coincide? Do second-order Skolem forms provide the only reasonable semantic interpretation for the syntax of partially ordered quantified formulas? The definition of branching quantifiers by generalized Skolem functions was propounded by Henkin, who recommended it as "natural." Most subsequent writers on the subject took Henkin's definition as given. I was led to reflect on the possibility of alternative definitions by J. Barwise's paper "On Branching Quantifiers in English" (1979). Barwise shifted the discussion from standard to generalized branching quantifiers, forcing us to rethink the principles underlying the branching structure. Reviewing the earlier controversy around Hintikka's purported discovery of branching quantifier constructions in natural language and following my own earlier inquiry into the nature of quantifiers, I came to think that both logico-philosophical and linguistic considerations suggest further investigation of the branching form.

2 Linguistic Motivation

In "Quantifiers vs. Quantification Theory" (1973), J. Hintikka first pointed out that some quantifier constructions in English are branching rather than linear. A well-known example is,

(10) Some relative of each villager and some relative of each townsman hate each other.[6]

Hintikka says, "This [example] may ... offer a glimpse of the ways in which branched quantification is expressed in English. Quantifiers occurring in conjoint constituents frequently enjoy independence of each other, it seems, because a sentence is naturally thought of as being symmetrical semantically vis-à-vis such constituents."[7] Another linguistic form of the branching-quantifier structure is illustrated by

(11) Some book by every author is referred to in some essay by every critic.[8]

Hintikka's point is that sentences such as (10) and (11) contain two *independent* pairs of iterated quantifiers, the quantifiers in each pair being outside the scope of the quantifiers in the other. A standard first-order formalization of such sentences—for instance, that of (10) as

(12) $(\forall x)(\exists y)(\forall z)(\exists w)(Vx \,\&\, Tz \rightarrow Ryx \,\&\, Rwz \,\&\, Hyw \,\&\, Hwy)$

or

(13) $(\forall x)(\forall z)(\exists y)(\exists w)(Vx \,\&\, Tz \rightarrow Ryx \,\&\, Rwz \,\&\, Hyw \,\&\, Hwy)$

(with the obvious readings for V, T, R, and H)—creates dependencies where none should exist. A branching-quantifier reading, on the other hand,

$$(14)\quad \begin{matrix}(\forall x)(\exists y)\\(\forall z)(\exists w)\end{matrix} \Big> Vx \,\&\, Tz \rightarrow Ryx \,\&\, Rwz \,\&\, Hyw \,\&\, Hwy,$$

accurately simulates the dependencies and independencies involved. Hintikka does not ask what truth conditions should be assigned to (14) but rather assumes that it is interpreted in the "usual" way as

(15) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\{Vx \,\&\, Tz \rightarrow R(f^1(x), x) \,\&\, R(g^1(z), z) \,\&$
$H(f^1(x), g^1(z)) \,\&\, H(g^1(z), f^1(x))\}$.

Hintikk'a paper brought forth a lively exchange of opinions, and G. Fauconnier (1975) raised the following objection (which I formulate in my own words): (15) implies that the relation of mutual hatred between relatives of villagers and relatives of townsmen has what we might call a *massive nucleus*—one that contains at least one relative of each villager and one relative of each townsman—and such that *each* villager relative in the nucleus hates *all* the townsman relatives in it, and vice versa. However, Fauconnier objects, it is not true that every English sentence with syntactically independent quantifiers implies the existence of a *massive nucleus* of objects standing in the quantified relation. For instance,

(16) Some player of every football team is in love with some dancer of every ballet company

does not.[9] It is compatible with the assumption that men are in love with one woman at a time (and that dancers/football-players do not belong to more than one ballet-company/football-team at a time). Even if Hintikka's interpretation of (10) is correct, Fauconnier continues, i.e., even if (10) implies the existence of a *massive nucleus* of villagers and townsmen in mutual hatred, (16) *does not* imply the existence of a *massive nucleus* of football players in love with dancers. Hintikka's interpretation, therefore, is not appropriate to all scopewise independent quantifiers in natural language. I illustrate the issue graphically in figures 5.1 and 5.2. The point is accentuated in the following examples:

(17) Some player of every football team is *the* boyfriend of some dancer of every ballet company.

(18) Some relative of each villager and some relative of each townsman are married (to one another).

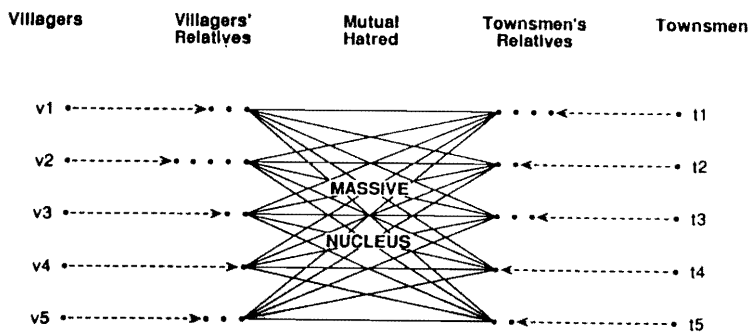

Figure 5.1

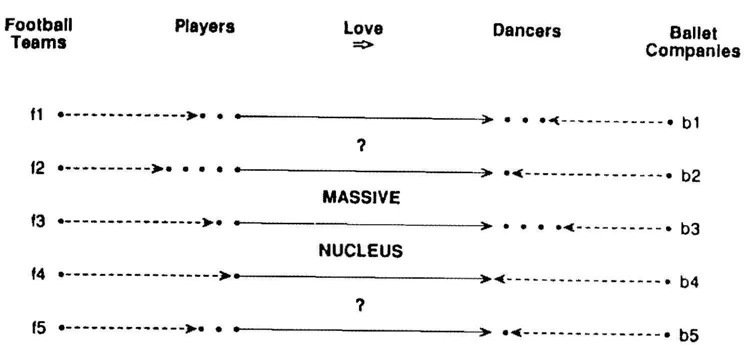

Figure 5.2

Is (17) logically false? Does (18) imply that the community in question is polygamous?

Fauconnier's conclusion is that natural-language constructions with quantifiers independent in scope are sometimes branching and sometimes linear, depending on the context. The correct interpretation of (16), for instance, is

(19) $(\forall x)(\forall y)(\exists z)(\exists w)(Fx \,\&\, By \rightarrow Pzx \,\&\, Dwy \,\&\, Lzw)$.

Thus, according to Fauconnier, the only alternative to "massive nuclei" is linear quantification.

We can, however, approach the matter somewhat differently. Acknowledging the semantic independence of syntactically unnested quantifiers in general, we can ask, Why should the independence of quantifiers have anything to do with the existence of a "massive nucleus" of objects standing in the quantified relation? Interpreting branching quantifiers nonlinearly, yet without commitment to a "massive nucleus," would do justice both to Hintikka's insight regarding the nature of scope-independent quantifiers and to Fauconnier's (and others') observations regarding the multiplicity of situations that such quantifiers can be used to describe. We are thus led to search for an alternative to Henkin's definition that would avoid the problematical commitment.

3 Logico-philosophical Motivation

Why are quantifier prefixes in modern symbolic logic linearly ordered? M. Dummett (1973) ascribes this feature of quantification theory to the genius of Frege. Traditional logic failed because it could not account for the validity of inferences involving multiple quantification. Frege saw that the problem could be solved if we construed multiply quantified sentences as complex *step-by-step* constructions, built by repeated applications of the simple logical operations of universal and/or existential quantification. This step-by-step syntactic analysis of multiply quantified sentences was to serve as a basis for a corresponding step-by step semantic analysis that unfolds the truth conditions of one constructional stage, i.e., a *singly quantified formula*, at a time. (See section 1 above.) In other words, by Frege's method of logical analysis the problem of defining truth for a quantified many-place relation was reduced to that of defining truth for a series of quantified predicates (1-place relations), a problem whose solution was essentially known.[10] The possibility of such a reduction was based, however, on a particular way of representing relations. In Tarskian semantics this form of representation is reflected in the way in which the linear steps in the definition of truth are "glued" together, namely by a relative expression synonymous with "for each one of which" ("f.e.w."). Thus, for example, the Fregean-Tarskian definition of truth for

(20) $(Q_1 x)(Q_2 y)(Q_3 z) R^3(x, y, z)$,

where Q_1, Q_2, and Q_3 are either $\forall$ or $\exists$, proceeds as follows: (20) is true in a model $\mathfrak{A}$ with a universe A *iff* there are $q_1 a$'s in A, f.e.w. there are $q_2 b$'s in A, f.e.w. there are q_3 c's in A such that "$R^3(a, b, c)$" is true in $\mathfrak{A}$, where q_1, q_2, and q_3 are the quantifier conditions associated with Q_1, Q_2, and Q_3 respectively.[11]

Intuitively, the view of R^3 embedded in the definition of truth for (20) is that of a *multiple tree*. (See figure 5.3.) Each row in the multiple tree represents one domain of R^3 (the extension of one argument place of R^3); each tree represents the restriction of R^3 to some one element of the domain listed in the upper row. In this way the extension of the second domain is represented *relative* to that of the first, and the extension of the third relative to the (already relative) representation of the second. Different quantifier prefixes allow different multiple-tree views of relations, but Frege's linear quantification limits the expressive power of quantifier prefixes to properties of relations that are discernible in a mutliple-tree representation.

We can describe the sense in which (all but the outermost) quantifiers in a linear prefix are semantically dependent as follows: a *linearly dependent* quantifier assigns a property not to a complete domain of the relation quantified but to a domain relativized to individual elements of another domain higher up in the multiple tree. It is characteristic of a linear quantifier prefix that each quantifier (but the outermost) is directly dependent on exactly one other quantifier. I will therefore call linear quantifiers *unidependent*- or *simply dependent*.

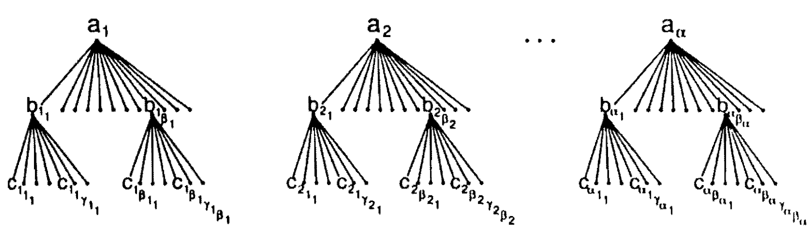

Figure 5.3

There are two natural alternatives to simple dependence: (1) no dependence, i.e., *independence*, and (2) *complex dependence*. These correspond to two ways in which we can view relations in a nonlinear manner: we can view each domain separately as complete and unrelativised, or we can view a whole cluster of domains at once in their mutual relationships.

Syntactically, I will represent an *independent* quantification by

$$(21)\quad \left.\begin{matrix}(Q_1 x_1)\\ \vdots\\ (Q_n x_n)\end{matrix}\right|\; R^n(x_1, \ldots, x_n)$$

and a *complex* quantification by

$$(22)\quad \begin{matrix}(Q_1 x_1)\\ \vdots\\ (Q_n x_n)\end{matrix} \Rightarrow R^n(x_1, \ldots, x_n).$$

Of course, there are many complex patterns of dependence among quantifiers. These can be represented by various partially ordered prefixes.

Our analysis indicates that the concept of independent quantification is different from that of complex quantification. Therefore, the first question regarding the correct interpretation of natural-language sentences with branching quantifiers is, Are the quantifiers in these sentences independent or complex?

4 Independent Branching Quantifiers

It is easy to give a precise definition of *independent quantification*:

$$(23)\quad \left.\begin{matrix}(Q_1 x)\\ \\ (Q_2 y)\end{matrix}\right|\; \Phi(x, y) =_{\mathrm{df}} (Q_1 x)(\exists y)\Phi(x, y) \;\&\; (Q_2 y)(\exists x)\Phi(x, y),$$

or more generally,

$$(24)\quad \left.\begin{matrix}(Q_1 x_1)\\ \vdots\\ (Q_n x_n)\end{matrix}\right|\; \Phi(x_1, \ldots, x_n) =_{\mathrm{df}}$$

$$(Q_1 x_1)(\exists x_2)\ldots(\exists x_n)\Phi(x_1, \ldots, x_n) \;\&\; \ldots \;\&$$
$$(Q_n x_n)(\exists x_1)\ldots(\exists x_{n-1})\Phi(x_1, \ldots, x_n).$$

This new definition of nonlinear quantification is very different from that of Henkin's. Independent quantification is essentially first-order. It does not involve commitment to a "massive nucleus" or to any other complex structure of objects standing in the quantified relation. Therefore, it enables us to analyze natural-language sentences with scope-independent quantifiers in a straightforward manner and without forcing any independent quantifier into a nested position. I thus propose (23) as a definition of branching quantifiers as *independent* quantifiers. Linguistically, this construal is supported by the fact that "and" often appears as a "quantifier connective" in natural-language branching structures in a way which might indicate a shift from its "original" position as a sentential connective. Moreover, natural-language branching quantifiers are symmetrical in much the same way that the conjuncts in my definition are. An English sentence with standard quantifiers that appears to exemplify independent quantification is

(25) Nobody loves nobody,

understood as "Nobody loves anybody."[12] I will symbolize (25) as

$$(26)\quad \begin{array}{l|l} (\sim\exists x) & \\ & Lxy \\ (\sim\exists y) & \end{array}$$

and interpret it as

(27) $\sim(\exists x)(\exists y)Lxy \;\&\; \sim(\exists y)(\exists x)Lxy.$

By extending our logical vocabulary to 1-place Mostowskian quantifiers, we will be able to interpret the following English sentences as independent branching quantifications:

(28) Three elephants were chased by a dozen hunters.

(29) Four Martians and five Humans exchanged insults.

(30) An odd number of patients occupied an even number of beds.

The "independent" interpretation of (28) to (30) reflects a "cumulative" reading, under which no massive nucleus, or any other complex relationship between the domain and the range of the relation in question, is intended.[13] We thus understand (28) as saying that the relation "elephant x was chased by hunter y" includes three individuals in its domain and a dozen individuals in its range. And this reading is captured by (23). Similarly, (23) yields the cumulative interpretations of (29) and (30).

The extension of the definition to 2-place Mostowskian quantifiers (which in this chapter I symbolize as Q^2 rather than $Q^{1,1}$) will yield independent quantifications of the form

$$(31)\quad \begin{array}{l|l} (Q_1^2 x) & \Psi_1 x, \\ & \qquad\qquad \Phi xy. \\ (Q_2^2 y) & \Psi_2 y, \end{array}$$

Here, however, we can apply the notion of independent quantification in several ways. Given a binary relation R, two sets A and B, and two quantifier conditions q_1 and q_2, we can say the following:

a. The relation R has q_1 As in its domain and q_2 Bs in its range.
b. The relation $A \upharpoonleft R \upharpoonright B$ has q_1 elements in its domain and q_2 elements in its range (where $A \upharpoonleft R \upharpoonright B$ is obtained from R by restricting its domain to A and its range to B).
c. The relation $A \upharpoonleft R \upharpoonright B$ has q_1 As in its domain and q_2 Bs in its range.
d. The relation $R \upharpoonright B$ has q_1 As in its domain and q_2 Bs in its range.

⋮

It is easy to see that (a) through (d) are not equivalent.[14] However, for the examples discussed here it suffices to define (31) for case (c). I thus propose as the definition of a pair of 2-place independent quantifiers

$$(32)\quad \begin{array}{l|l} (Q_1^2 x) & \Psi_1 x, \\ & \qquad \Phi xy =_{df} (Q_1^2 x)[\Psi_1 x, (\exists y)(\Psi_1 x \,\&\, \Psi_2 y \,\&\, \Phi xy)] \,\& \\ (Q_2^2 y) & \Psi_2 y, \qquad\qquad\qquad (Q_2^2 y)[\Psi_2 y, (\exists x)(\Psi_1 x \,\&\, \Psi_2 y \,\&\, \Phi xy)]. \end{array}$$

When Q_1^2 and Q_2^2 satisfy the property of *living on*, i.e., when $(Q^2 x)(\Phi x, \Psi x)$ is logically equivalent to $(Q^2 x)(\Phi x, \Phi x \,\&\, \Psi x)$, we can replace (32) with the simpler

$$(33)\quad \begin{array}{l|l} (Q_1^2 x) & \Psi_1 x, \\ & \qquad \Phi xy =_{df} (Q_1^2 x)[\Psi_1 x, (\exists y)(\Psi_2 y \,\&\, \Phi xy)] \,\& \\ (Q_2^2 y) & \Psi_2 y, \qquad\qquad\qquad (Q_2^2 y)[\Psi_2 y, (\exists x)(\Psi_1 x \,\&\, \Phi xy)]. \end{array}$$

Using this definition, we can interpret (34) and (35) below as independent quantifications:

(34) All the boys ate all the apples.[15]

(35) Two boys ate half the apples.

We can also analyze (28) to (30) as independent quantifications of the form (33).[16]

What about Hintikka's (10) and Fauconnier's (16)? Should we interpret these as independent branching quantifications of the form (33)? Under such an interpretation, (10) would say that the relation of mutual hatred between relatives of villagers and relatives of townsmen includes at least one relative of each villager in its domain and at least one relative of each townsman in its range; (16) would be understood as saying that the relation of love between football players and ballet dancers includes at least one player of each football team in its domain and at least one dancer from each ballet company in its range. Such interpretations would be compatible with both figures 5.1 and 5.2. Later on I will suggest a test to determine whether the intended interpretation of a given natural-language sentence with branching quantifiers is that of an independent or complex quantification, and this might give us a clue regarding Hintikka's and Fauconnier's sentences. As for the linear option, here the question is whether one pair of quantifiers is within the scope of the other. Generally, I would say that when "and" appears as a quantifier connective, that is, "Q_1 *As and* Q_2 *Bs* stand in relation *R*," the quantification is not linear. However, when the quantification is of the form "Q_1 *As R* Q_2 *Bs*," the situation is less clear. (For further discussion, see May 1989 and van Benthem 1989.[17]) I should note that sometimes the method of semantic representation itself favors one interpretation over another. For example, in standard semantics, relations are so represented that it is impossible for the range of a given binary relation to be empty when its domain is not empty. Thus a quantification of the form "Three *A*s stand in the relation *R* to zero *B*s" would be logically false if interpreted as independent branching quantification. To render it logically contingent, we may construe it as a nested quantification of two 1-place predicative quantifiers, and this gives us the linear reading.

5 Barwise's Generalization of Henkin's Quantifiers

I now turn to *complex quantification*. Evidently, Henkin's quantifiers belong in this category. I ask: What kind of information on a quantified relation does a complex quantifier prefix give us? As we shall soon see, the shift to a more general system of quantifiers, namely Mostowski's 1- and 2-place predicative quantifiers, throws a new light on the nature of complex branching quantification.

Barwise (1979) generalized Henkin's definition of standard branching quantifiers to 1-place monotone-increasing Mostowskian quantifiers in the following way:[18]

(36)
$$\begin{array}{l}(Q_1 x)\\ (Q_2 y)\end{array}\!\!> \Phi xy =_{df} (\exists X)(\exists Y)[(Q_1 x)Xx \,\&\, (Q_2 y)Yy \,\&\, (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow \Phi xy)].$$[19]

Technically, the generalization is based on a relational reading of the Skolem functions in Henkin's definition. Thus, Barwise's equivalent of Henkin's (8) is

(37)
$$(\exists R)(\exists S)[(\forall x)(\exists y)Rxy \,\&\, (\forall z)(\exists w)Szw \,\&\, (\forall x)(\forall y)(\forall z)(\forall w)(Rxy \,\&\, Szw \rightarrow \Phi(x, y, z, w))].$$

Clearly, Barwise's quantifiers, like Henkin's, are *complex*, not independent, branching quantifiers.

Barwise suggested that this generalization enables us to give English sentences with unnested monotone-increasing generalized quantifiers a "Henkinian" interpretation similar to Hintikka's interpretation of (10) and (11). Here are two of his examples:[20]

(38) Most philosophers and most linguists agree with each other about branching quantification.

(39) Quite a few boys in my class and most girls in your class have all dated each other.

To interpret (38) and (39), we have to extend (36) to 2-place predicative quantifiers. This we do as follows: Let Q_1^2 and Q_2^2 be 2-place monotone-increasing predicative quantifiers. Then

(40)
$$\begin{array}{l}(Q_1^2 x) \cdot \Psi_1 x,\\ (Q_2^2 y) \cdot \Psi_2 y,\end{array}\!\!> \Phi xy =_{df} (\exists X)(\exists Y)[(Q_1^2 x)(\Psi_1 x, Xx) \,\&\, (Q_2^2 y)(\Psi_2 y, Yy) \,\&\, (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow \Phi xy)].$$

We can now interpret (38) as

(41)
$$\begin{array}{l}(M^2 x) \cdot Px,\\ (M^2 y) \cdot Ly,\end{array}\!\!> Axy \,\&\, Ayx =_{df} (\exists X)(\exists Y)[(M^2 x)(Px, Xx) \,\&\, (M^2 y)(Ly, Yy) \,\&\, (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow Axy \,\&\, Ayx)],$$

with the obvious readings of *P*, *L*, *A* and where "M^2" stand for the 2-place "most." We interpret (39) in a similar manner.

Barwise emphasized that his definition of branching monotone-increasing generalized quantifiers is not applicable to monotone-decreasing, non-monotone, or mixed branching quantifiers.[21] This is easily explained by the absurd results of applying (36) to such quantifiers: (36) would render any monotone-decreasing branching formula vacuously true (by taking *X* and *Y* to be the empty set); it would render false non-monotone branching formulas true, as in the case of "Exactly one *x* and exactly one *y* stand in the relation *R*," where *R* is universal and the cardinality of the universe is larger than 1.

Barwise proposed the following definition for a pair of 1-place monotone-decreasing branching quantifiers:

$$(42)\quad \begin{matrix}(Q_1 x)\\ \\ (Q_2 y)\end{matrix} > \Phi xy =_{df} (\exists X)(\exists Y)[(Q_1 x)Xx \mathbin{\&} (Q_2 y)Yy \mathbin{\&} (\forall x)(\forall y)(\Phi xy \to Xx \mathbin{\&} Yy)].$$ [22]

Definition (42), or its counterpart for 2-place quantifiers, provides an intuitively correct semantics for English sentences with a pair of unnested monotone-decreasing quantifiers. Consider, for instance,

(43) Few philosophers and few linguists agree with each other about branching quantification.

As to non-monotone and mixed branching quantifiers, Barwise left the former unattended and skeptically remarked about the latter, "There is no sensible way to interpret

$$(\mathrm{s})\quad \begin{matrix}Q_1 x\\ \\ Q_2 y\end{matrix} > A(x, y)$$

when one [quantifier] is increasing and the other is decreasing. Thus, for example,

(t) ?Few of the boys in my class and most of the girls in your class have all dated each other.

appears grammatical, but it makes no sense."[23]

Barwise's work suggests that the semantics of branching quantifiers depends on the monotonic properties of the quantifiers involved. The truth conditions for a sentence with branching monotone-increasing quantifiers are altogether different from the truth conditions for a sentence with branching monotone-decreasing quantifiers, and truth for sentences with mixed branching quantifiers is simply undefinable. Is the meaning of branching quantification as intimately connected with monotonicity as Barwise's analysis may lead one to conclude?

First, I would like to observe that Barwise interprets branching monotone-decreasing quantifiers simply as *independent* quantifiers: when Q_1 and Q_2 are monotone-decreasing (42) is logically equivalent to my (23). The latter definition, as we have seen, has meaning—the same meaning—for all quantifiers, irrespective of monotonicity. On this first-order reading, (43) says that the relation of mutual agreement about branching quantification between philosophers and linguists includes (at most) few philosophers in its domain and (at most) few linguists in its range.

Barwise explained the limited applicability of (36) in the following way: Every formula of the form

(44) $(Qx)\Phi x$,

where Q is monotone-increasing, is logically equivalent to a second-order formula of the form

(45) $(\exists X)[(Qx)Xx \mathbin{\&} (\forall x)(Xx \to \Phi x)]$,

which is structurally similar to (36). This fact establishes (36) as the correct definition of branching monotone-increasing quantifiers. However, (45) is not a second-order representation of quantified formulas with non-monotone-increasing quantifiers. Hence (36) does not apply to branching quantifiers of the latter kind. The definition of branching monotone-decreasing quantifiers by (42) is explained in a similar manner: when Q is monotone-decreasing, (44) is logically equivalent to

(46) $(\exists X)[(Qx)Xx \mathbin{\&} (\forall x)(\Phi x \to Xx)]$,

which is structurally similar to (42).[24]

I do not find this explanation convincing. Linear quantifiers vary with respect to monotonicity as much as branching quantifiers do, yet the semantic definition of linear quantifiers is the same for *all* quantifiers, irrespective of monotonicity. Linear quantification is also meaningful for all *combinations* of quantifiers. Why should the meaningfulness of the branching form stop short at mixed monotone quantifiers? Moreover, if the second-order representation of "simple" first-order quantifications determines the correct analysis of branching quantifications, Barwise has not shown that there is no second-order representation of (44) that applies *universally*, without regard to monotonicity.

6 A General Definition of Complex, Henkin-Barwise Branching Quantifiers

The conception of complex branching quantification embedded in Barwise's (36) assigns the following truth conditions to branching formulas of the form

$$(47)\quad \begin{matrix}(Q_1 x)\\ \\ (Q_2 y)\end{matrix} > \Phi xy,$$

where Q_1 and Q_2 are monotone-increasing:

DEFINITION 1 The branching formula (47) is true in a model $\mathfrak{A}$ with universe A iff there is at least one pair, $\langle X, Y\rangle$, of subsets of A for which the following conditions hold:

1. X satisfies the quantifier condition Q_1.
2. Y satisfies the quantifier condition Q_2.
3. *Each* element of X stands in the relation $\Phi^{\mathfrak{A}}$ to *all* the elements of Y.

The condition expressed by (3) I shall call the *each-all* (or *all-all*) condition on $\langle X, Y\rangle$ with respect to $\Phi^{\mathfrak{A}}$. We can then express definition 1 more succinctly as follows:

DEFINITION 2 The branching formula (47) is true in a model $\mathfrak{A}$ with a universe A iff there is at least one pair of subsets of the universe satisfying the *each-all* condition with respect to $\Phi^{\mathfrak{A}}$, with its first element satisfying Q_1 and its second element satisfying Q_2.

Set-theoretically, definition 2 says that $\Phi^{\mathfrak{A}}$ includes at least one *Cartesian product* of two subsets of the universe satisfying Q_1 and Q_2 respectively. (The "massive nucleus" of section 2 above was an informal term for a Cartesian product.)

Is the complex quantifier condition expressed by definition 2 meaningful only with respect to monotone-increasing quantifiers? I think that the idea behind this condition makes sense no matter what quantifiers Q_1 and Q_2 are. However, this idea is not adequately formulated in definition 2 as it now stands, since this definition fails to capture the intended condition when Q_1 and/or Q_2 are not monotone-increasing. In that case Q_1 and/or Q_2 set a limit on the size of sets X and/or Y such that $\langle X, Y\rangle$ satisfies the *each-all* condition with respect to $\Phi^{\mathfrak{A}}$: (47) is true only if a Cartesian product small enough or of a particular size is included in $\Phi^{\mathfrak{A}}$. But definition 2 in its present form cannot express this condition: if $\Phi^{\mathfrak{A}}$ includes a Cartesian product larger than required, definition 2 is automatically satisfied. This is because for any two nonempty sets A and B, if $A \times B$ is a Cartesian product included in $\Phi^{\mathfrak{A}}$, so is $A' \times B'$, where A' and B' are any proper subsets of A and B respectively. The difficulty, however, appears to be purely technical. We can overcome it by demanding that the condition be met by a *maximal*, not a sub-, Cartesian product. In other words, only maximal Cartesian products included in $\Phi^{\mathfrak{A}}$ should count as satisfying the *each-all* condition.

I thus add a *maximality condition* to definition 1 and arrive at the following general definition, in which no restrictions are set on Q_1 and Q_2:

DEFINITION 3 The branching formula (47) is true in a model $\mathfrak{A}$ with universe A iff there is at least one pair $\langle X, Y\rangle$ of subsets of A for which

the following conditions hold:

1. X satisfies quantifier condition Q_1.
2. Y satisfies quantifier condition Q_2.
3. Each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to all the elements of Y.
4. The pair $\langle X, Y\rangle$ is a maximal pair satisfying (3).

Referring to (3) and (4) as "the *maximal each-all* condition on $\langle X, Y\rangle$ with respect to Φ," we can reformulate definition 3 more concisely as follows:

DEFINITION 4 The branching formula (47) is true in a model $\mathfrak{A}$ with universe A iff there is at least one pair of subsets in the universe satisfying the *maximal each-all* condition with respect to $\Phi^{\mathfrak{A}}$ such that its first element satisfies Q_1 and its second element satisfies Q_2.

I thus propose to replace (36) with

$$(48)\qquad \begin{array}{l}(Q_1 x)\searrow\\ \qquad\qquad \Phi xy =_{df}\\ (Q_2 y)\nearrow\end{array}$$

$$\begin{aligned}(\exists X)(\exists Y)\{(Q_1 x)Xx \,\&\, (Q_2 y)Yy \,\&\, (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow \Phi xy) \,\&\,\\ (\forall X')(\forall Y')[(\forall x)(\forall y)((Xx \,\&\, Yy \rightarrow X'x \,\&\, Y'y) \,\&\,\\ (X'x \,\&\, Y'y \rightarrow \Phi xy)) \rightarrow\\ (\forall x)(\forall y)(Xx \,\&\, Yy \leftrightarrow X'x \,\&\, Y'y)]\}\end{aligned}$$

as the definition of Henkin-Barwise complex branching quantifiers. We can rewrite (48) more succinctly, using common conventions, as

$$(49)\qquad \begin{array}{l}(Q_1 x)\searrow\\ \qquad\qquad \Phi xy =_{df}\\ (Q_2 y)\nearrow\end{array}$$

$$\begin{aligned}(\exists X)(\exists Y)[(Q_1 x)Xx \,\&\, (Q_2 y)Yy \,\&\, X \times Y \subseteq \Phi \,\&\,\\ (\forall X')(\forall Y')(X \times Y \subseteq X' \times Y' \subseteq \Phi \rightarrow X \times Y = X' \times Y')].\end{aligned}$$

More concisely yet, we have

$$(50)\qquad \begin{array}{l}(Q_1 x)\searrow\\ \qquad\qquad \Phi xy =_{df}\\ (Q_2 y)\nearrow\end{array}$$

$$\begin{aligned}(\exists X)(\exists Y)[(Q_1 x)Xx \,\&\, (Q_2 y)Yy \,\&\,\\ (\forall X')(\forall Y')(X \times Y \subseteq X' \times Y' \subseteq \Phi \leftrightarrow X \times Y = X' \times Y')].\end{aligned}$$

It is easy to see that whenever Q_1 and Q_2 are monotone-increasing, (49) is logically equivalent to (36). At the same time, (49) avoids the problems that arise when (36) is applied to non-monotone-increasing quantifiers.

Maximality conditions are very common in mathematics. Generally, when a structure is maximal, it is "complete" in some relevant sense.[25] The Henkin-Barwise branching quantifier prefix expresses a certain condition on sets (subsets of the quantified relation). And when we talk about sets, it is usually maximal sets that we are interested in. Indeed, conditions on sets are normally conditions on maximal sets. Consider, for instance, the statement "Three students passed the test." Would this statement be true had 10 students passed the test? But it would be if the quantifier "!3" set a condition on a nonmaximal set: a partial extension of "x is a student who passed the test" would satisfy that condition. Consider also "No student passed the test" and "Two people live in America."

The fact that quantification in general sets a condition on maximal sets (relations) is reflected by the equivalence of any first-order formula of the form

(44) $(Qx)\Phi x$,

no matter what quantifier Q is (monotone-increasing, monotone-decreasing or non-monotone), to

(51) $(\exists X)[(Qx)Xx \ \& \ X \subseteq \Phi \ \& \ (\forall X')(X \subseteq X' \subseteq \Phi \rightarrow X' = X)]$,

which expresses a maximality condition. The logical equivalence of (44) to (51) provides a further justification for the reformulation of (36) as (49).

We have seen that the two conceptions of nonlinear quantification discussed so far, *independence* (first-order) and *complex dependence* (second-order), have little to do with monotonicity or its direction. The two conceptions lead to entirely different definitions of the branching quantifier-prefix, both, however, universally applicable.

Linguistically, my suggestion is that to determine the truth conditions of natural-language sentences with a nonlinear quantifier-prefix, one has to ask not whether the quantifiers involved are monotone-increasing, monotone-decreasing, etc. but whether the prefix is independent or complex. My analysis points to the following clue: Complex Henkin-Barwise quantifications always include an inner *each-all* condition, explicit or implicit. Independent quantifications, on the other hand, do not include any such condition.

Barwise actually gave several examples of branching sentences with an explicit *each-all* condition:

(39) Quite a few boys in my class and most girls in your class have *all* dated each other.[26]

(52) Most of the dots and most of the stars are *all* connected by lines.[27]

Such an explicit "all" also appears in his

(t) Few of the boys in my class and most of the girls in your class have *all* dated each other.[28]

I therefore suggest that we interpret Barwise's (t) as an instance of (49).

Some natural examples of Henkin-Barwise complex branching quantifiers in English involve non-monotonic quantifiers. For example,

(53) A couple of boys in my class and a couple of girls in your class were *all* dating each other.

(54) An even number of dots and an odd number of stars are *all* connected by lines.

Another expression that seems to point to a complex branching structure (which indicates a second-order form) is "the same." Consider

(55) Most of my friends have applied to the *same* few graduate programs.

To interpret the above sentences accurately, we have to extend (49) to 2-place quantifiers. As in the case of 2-place independent quantifiers (see section 4 above), we can apply the notion of complex *each-all* quantification in more than one way. I will limit my attention to one of these, defining "Q_1 As and Q_2 Bs all stand in the relation R" as "There is at least one maximal Cartesian product included in $A \upharpoonleft R \upharpoonright B$ with Q_1 As in its domain and Q_2 Bs in its range." In symbols,

$$(56)\quad \begin{matrix} (Q_1^2 x) \cdot \Psi_1 x, \\ \cdot \\ (Q_2^2 y) \cdot \Psi_2 y, \end{matrix} \quad \Phi xy =_{df}$$

$$(\exists X)(\exists Y)[(Q_1^2 x)(\Psi_1 x, Xx) \ \& \ (Q_2^2 y)(\Psi_2 y, Yy) \ \& \ (\forall X')(\forall Y')(X \times Y \subseteq X' \times Y' \subseteq \Psi_1 \upharpoonleft \Phi \upharpoonright \Psi_2 \leftrightarrow X \times Y = X' \times Y')].$$

Linguistically, my account explains the meaning (function) of inner quantifiers that, like Barwise's "all," do not bind any new individual variables in addition to those bound by Q_1 and Q_2. A "standard" reading of such quantifiers is problematic, since all the variables are already bound by the outer quantifiers. On my analysis, these quantifiers point to a second-order condition.

Going back to the controversy regarding Hintikka's reading of natural-language sentences with symmetrical quantifiers, we can reformulate Fauconnier's criticism as follows: Some natural-language sentences with unnested quantifiers do not appear to contain, explicitly or implicitly, an inner *each-all* quantifier condition. On my analysis, these are not Henkin-Barwise branching quantifications. Whether Hintikka's (10) includes an implicit *each-all* condition, I leave an open question. (One way to justify Hintikka's claim that (10) is a Henkin sentence is to interpret "each" in "each other" as elliptic for "each-all.")

The reading of a natural-language branching quantification with no explicit *each-all* condition involves various linguistic considerations. Our logical point of view has so far indicated three possible readings: as an independent quantification, as a linear quantification, or as a Henkin-Barwise complex quantification. But as we will presently see, these are not the only options. In the next section I will introduce a "family of interpretations" that extends considerably the scope of nonlinear quantification.

7 Branching Quantifiers: A Family of Interpretations

The Henkin-Barwise definition of branching quantifiers, in its narrow as well as general form, includes two quantifier conditions in addition to those explicit in the definiendum: the outer quantifier condition "there is at least one pair $\langle X, Y\rangle$" and the inner (maximal) *each-all* quantifier condition. By generalizing these conditions, we arrive at a new definition schema whose instances comprise a family of semantic interpretations for multiple quantifiers. Among the members of this family are both the independent branching quantifiers of section 4 and the Henkin-Barwise complex quantifiers of section 6. This generalized definition schema delineates a totality of *forms of quantifier dependence*. Degenerate dependence is independence; linear dependence is a particular case of (non-degenerate) Henkin-Barwise dependence.[29]

We arrive at the definition schema in two steps. First we generalize the inner *each-all* quantifier condition (see definitions 1–4), and we obtain the following schema:

GENERALIZATION 1 A branching formula of the form (47) is true in a model $\mathfrak{A}$ with a universe A iff for at least one pair $\langle X, Y\rangle$ of subsets of the universe satisfying the maximal quantifier condition $\mathcal{Q}_1$ with respect to $\Phi^{\mathfrak{A}}$, X satisfies Q_1, and Y satisfies Q_2,

where $\mathcal{Q}_1$ represents any (first-order) maximal quantifier-condition on a pair of subsets of the universe with respect to $\Phi^{\mathfrak{A}}$. The following are a few instances of $\mathcal{Q}_1$:

Condition A: one one The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to exactly one element of Y and for each element of Y there is exactly one element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Condition B: each two or more The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to two or more elements of Y and for each element of Y there is an element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Condition C: each more than ___ The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to more than ___ elements of Y and for each element of Y there is an element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Condition D: each at least half/at least half-each The pair $\langle X, Y\rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to at least half the elements of Y and to each element of Y at least half the elements of X stand in the relation $\Phi^{\mathfrak{A}}$.

We can find natural-language sentences that exemplify generalization 1 by substituting conditions A through D for $\mathcal{Q}_1$:

(57) Most of my right-hand gloves and most of my left-hand gloves match (one to one).

(58) Most of my friends saw at least two of the same few Truffaut movies.

(59) The same few characters repeatedly appear in many of her early novels.

(60) Most of the boys and most of the girls in this party are such that each boy has chased at least half the girls and each girl has been chased by at least half the boys.

The adaptation of generalization 1 to 2-place quantifiers, needed in order to give these sentences precise interpretations, is analogous to (56).

We can verify the correctness of our interpretations by checking whether (57) to (60) can be put in the following canonical forms:

(61) Most of my right-hand gloves and most of my left-hand gloves

are such that each of the former matches exactly one of the latter and vice versa.

(62) Most of my friends and few of Truffaut's movies are such that each of the former saw at least two of the latter and each of the latter was seen by at least one of the former.

(63) Few characters and many of her early novels are such that each of the former appears in more than __ of the latter and each of the latter includes at least one of the former.

Sentence (60) is already in canonical form.

By replacing $\mathscr{Q}_1$ in generalization 1 with condition E below we get the *independent quantification* of section 6.

Condition E: each–some/some–each The pair $\langle X, Y \rangle$ is a maximal pair such that each element of X stands in the relation $\Phi^{\mathfrak{A}}$ to some element of Y and for each element of Y there is some element of X that stands to it in the relation $\Phi^{\mathfrak{A}}$.

Thus, both independent branching quantifiers and complex, Henkin-Barwise branching quantifiers fall under the general schema.

The second generalization abstracts from the outermost existential condition:

GENERALIZATION 2 A branching formula of the form (47) is true in a model $\mathfrak{A}$ with universe A iff there are $\mathscr{Q}_2$ pairs $\langle X, Y \rangle$ of subsets of the universe satisfying the maximal quantifier condition $\mathscr{Q}_1$ with respect to $\Phi^{\mathfrak{A}}$ such that X satisfies Q_1 and Y satisfies Q_2.[30]

The following sentences exemplify generalization 2 by substituting "by and large" (interpreted as "most") and "at most few" for $\mathscr{Q}_2$ ($\mathscr{Q}_1$ is the *each–all* condition):

(64) *By and large*, no more than a few boys and a few girls all date one another.

(65) There are *at most few* cases of more than a couple Eastern delegates and more than a couple Western delegates who are all on speaking terms with one another.[31]

The family of branching structures delineated above enlarges considerably the array of interpretations available for natural-language sentences with multiple quantifiers. The task of selecting the right alternative for a given natural-language quantification is easier if explicit inner and outer quantifier conditions occur in the sentence, but is more complicated other-

wise. One could, of course, be assisted by "context," but linguists will be interested in formulating general guidelines that hold across contexts. Indeed, we may look at Barwise's claims regarding monotone-increasing and monotone-decreasing English branching quantifiers in this light. According to Barwise, in English monotone-increasing branching quantifiers are usually accompanied by an inner "all," indicating a complex "*each all* structure" (with "some" as the outer quantifier condition); monotone-decreasing quantifiers are usually not accompanied by an inner quantifier condition, pointing to an independent (*each–some/some–each*) structure. These conjectures can be expressed in terms of my general definition schema of branching quantification (generalization 2).[32] However, the new multiplicity of inner and outer quantifier conditions introduced in the present section calls for refinement and supplementation of Barwise's conjectures.

8 Conclusion

My investigation has yielded a general definition schema for a pair of branching, or partially ordered, generalized quantifiers. The existing definitions, due to Barwise, constitute particular instances of this schema. The next task is to extend the schema, or particular instances thereof, especially (49), to arbitrarily large partially ordered quantifier prefixes. This task, however, is beyond the scope of the present work.

In "Branching Quantifiers and Natural Language" (1987), D. Westerståhl proposed a general definition of (Barwise's) branching quantifiers different from the ones suggested here. Although Westerståhl's motivation was similar to mine (dissatisfaction with the multiplicity of partial definitions), he approached the problem in a different way. Accepting Barwise's definitions of monotone-increasing and monotone-decreasing branching quantifiers, along with van Benthem's definition of branching non-monotonic quantifiers of the form "exactly n," Westerståhl constructed a general formula that yields the above definitions when the quantifiers plugged in have the "right" kind of monotonicity. That is, Westerståhl was looking for an umbrella under which the various partial existent definitions would fall. From the point of view of the issues discussed here, Westerståhl's approach is very similar to Barwise's. For that reason I did not include a separate discussion of his approach.[33] As for van Benthem's proposal for the analysis of non-monotonic branching quantifiers, his definition is

$$(66)\quad \begin{array}{l}(\text{Exactly-}n\,x)\cdot Ax,\\ \qquad\qquad\cdot\\ (\text{Exactly-}m\,y)\cdot By,\end{array}\quad Rxy =_{\mathrm{df}} (\exists X)(\exists Y)(X \subseteq A \;\&\; Y \subseteq B \;\&\; |X| = n \;\&\; |Y| = m \;\&\; R = X \times Y).$$[34]

For 1-place quantifiers, the definition would be

$$(67)\quad \begin{array}{l}(\text{Exactly-}n\,x)\\ (\text{Exactly-}m\,y)\end{array}\Big> Rxy =_{\mathrm{df}} (\exists X)(\exists Y)(|X| = n \;\&\; |Y| = m \;\&\; R = X \times Y).$$

Since (67) is equivalent to (68) when R is not empty, I can express van Benthem's proposal in terms of my second generalization by saying that quantifiers of the form "exactly n" tend to occur in complex quantifications in which $\mathcal{Q}_1$ is "each–all" and $\mathcal{Q}_2$ is "the (only)."

(68) *The* (*only*) pair $\langle X, Y\rangle$ of subsets of the universe satisfying the *maximal each-all* condition with respect to R is such that X has *exactly n* elements and Y has *exactly m* elements.

I would like to end with a few general notes. Russell, recall, divided the enterprise of logic into two parts: the discovery of universal "templates" of truth and the discovery of new, philosophically significant logical forms. Branching quantifiers offer a striking example of an altogether new logico-linguistic form unlike anything thought to belong to language before Henkin's paper. One cannot, however, avoid asking: When does a generalization of a particular linguistic structure lead to a new, more general form of *language* and when does it end in a formal system that can no longer be considered language? Henkin, for instance, mentioned the possibility of constructing a densely ordered quantifier prefix. Would this be considered language? What about a prefix of quantifiers organized in some non-ordering pattern? Even the thoroughly studied form of an infinitely long linear prefix has yet to be evaluated with respect to our general concept of language.

Another question concerns the possibility of "importing" new structures into natural language. New forms continuously "appear" in all branches of mathematics and abstract logic. The "discovery" of branching prefixes in English makes one wonder whether new constructions cannot be introduced into natural language as well. Let us look back at Hintikka's "revelation" that branching quantifiers exist in English. Did Hintikka discover that all along we were talking about villagers' and townsmen's relatives hating each other *en masse* (each-all hatred) when we said that some relative of each villager and some relative of each townsman hate each other? Or did he, perhaps, propose to give a new meaning to a

syntactically well-formed but semantically empty (loosely defined) linguistic form? I am not sure what the right answer to this question is. Some of the English examples discussed in the literature strike me as having had a clear branching meaning even before the official seal of "branching quantification" was affixed to them. But others impress me as having been hopelessly vague before the advent of branching theory. These could have been semantically undetermined structures, forms in quest of content. Present-day languages have not used up all their lexical resources. Is logical form another unexhausted resource?

Investigations of the branching structure in the context of "generalized" logic led Barwise to extend Henkin's theory. My own inquiries have led to an even broader approach. In the next chapter I will return to the general conception of logic developed in this book and introduce some of its philosophical consequences. The philosophical ramifications of "unrestricted" logic have never before been (publicly) investigated. I will briefly point the direction of some philosophical inquiries and spell out a few results.

Chapter 6
A New Conception of Logic

The broad questions underlying this work concern the scope and limits of logic. Are the principles underlying modern logic fully exhausted by the standard system? Do generalized quantifiers signify a genuine breakthrough in logic? What are the boundaries of logic from the point of view of modern semantics? Starting with a general outlook of logic, I proceeded to examine Mostowski's generalization of the standard quantifiers, tracing its origins to Frege's interpretation of number statements. I then used Mostowski's theory as a jumping board for investigating the notion of "logicality." The initially loose philosophical question regarding the principles of logic received specific content: What makes a linguistic expression into a logical term? What are all the logical terms? My method of answering this question was conceptual. Examining Tarski's foundational work in semantics, I was able to identify a central motivation for constructing logic as a syntactic-semantic system in which logical truths and consequences are determined by reference to a full-blown system of models. I showed that within the framework of model-theoretic semantics the success of the logical project depends on the choice of logical terms. Inasmuch as logical constants represent the formal and necessary constituents of possible states of affairs, the system will accomplish its task. But the task is fully accomplished only if all formal and necessary constituents are taken into account. The standard system carries us one step toward the goal. It takes the full range of Tarskian or first-order Unrestricted Logic (UL) to achieve the objective in full. This outlook on logic is realized by logicians working within the dynamic field called "abstract" logic. It is also reflected in the work of linguists seeking to enhance the resources for studying the logical structure of natural language.

If the central claim of this book is correct, namely that standard mathematical logic, with its limited set of logical constants, does not fully express

the idea of logic, the question arises of whether a conceptual revision in the "official" doctrine is called for. Should "unrestricted logic" become "standard" logic? Because of the prominent place of standard first-order logic not only in mathematics but also in philosophy, linguistics, and related disciplines, at stake is a change in a very general and basic conceptual scheme. What are the philosophical ramifications of the new conception of logic? What new light does it shed on old philosophical questions? Are the conditions ripe for an "official" revision? And how should the new developments in semantics be viewed from the standpoint of proof theory? I would like to end this work with reflections on some aspects of these questions.

1 Revision in Logic

Putnam has convincingly argued that a change in a deeply ingrained conceptual scheme is seriously entertainable only if a well-developed alternative already exists. Referring to the revolution in geometry, Putnam argued that the laws of Euclidean geometry could not have been abandoned "before someone had worked out non-Euclidean geometry. That is to say, it is inconceivable that a scientist living in the time of Hume might have come to the conclusion that the laws of Euclidean geometry are false: 'I do not know what geometrical laws are true but I know the laws of Euclidean geometry are false.'"[1] Principles at the very center of our conceptual system are not overthrown unless "a rival theory is available."[2]

Is there a serious alternative to standard logical theory incorporating the principles of Unrestricted Logic delineated in this book? The unequivocal answer is yes. There exists a rich body of literature, in mathematics as well as in linguistics, in which nonstandard systems of first-order logic satisfying (UL) have been developed, studied, and applied. Mostowski's and Lindström's pioneering work led to a surge of logico-mathematical research. From Lindström's famous characterizations of "elementary logic" (1969) to works like Keisler's proof of the completeness of first-order logic with the quantifier "there exist uncountably many," the yield of mathematical investigations is astounding. For a representative collection of articles plus a comprehensive bibliography of more than a thousand items, the reader is referred to the 1985 volume *Model-Theoretic Logics*, edited by Barwise and Feferman.

In linguistics, Barwise and Cooper's 1981 paper also led to a profusion of literature. Generalized quantifiers became an essential component of formal semantics and of the theory of Logical Form within generative grammar. Representative works by van Benthem, Keenan, May, Westerståhl, and others are listed in the references.

Philosophically, the view that Tarskian or unrestricted logic is logic proper has not yet received its due attention. In particular, few philosophical arguments in support of the new view appeared in print. I hope this book has, to some extent, filled the void. If my argument is cogent, it will add a new philosophical dimension to the support that the new logic has received from other quarters.

2 The Logicist Thesis

The logicist thesis says that mathematics is reducible to logic in the sense that all mathematical theories can be formulated by purely logical means. That is, all mathematical constants are definable in terms of logical constants and all the theorems of (classical) mathematics are derivable from purely logical axioms by means of logical rules of derivation (and definitions). Now for the logicist thesis to be meaningful, the notions of logical constant, logical axiom, logical rule of derivation, and definition must be well defined and, moreover, so defined as to make the reduction nontrivial. In particular, it is essential that the reduction of mathematics to logic be carried out relative to a system of logic in which mathematical constants do not, in general, appear as primitive logical terms. The "fathers" of logicism did not engage in a critical examination of the concept of logical constant from this point of view. That is, they took it for granted that there is a small group of constants in terms of which the reduction is to be carried out: the truth-functional connectives, the existential (universal) quantifier, identity, and possibly the set-membership relation. The new conception of logic, however, contests this assumption. If my analysis of the semantic principles underlying modern logic in chapter 3 is correct, then any mathematical predicate or functor satisfying condition (E) can play the role of a primitive logical constant. Since mathematical constants in general satisfy (E) when defined as higher-level, the program of reducing mathematics to logic becomes trivial. Indeed, even if the whole of mathematics could be formulated within pure standard first-order logic, then (since the standard logical constants are nothing more than certain particular mathematical predicates) all that would have been accomplished is a reduction of some mathematical notions to others.

While the logicist program is meaningless from the point of view of the new conception of logic, its main tenet, that mathematical constants are essentially logical, is, of course, strongly supported by this conception.

Indeed, Russell's account of the logicality of mathematics in *Introduction to Mathematical Philosophy* is in complete agreement with my analysis:

> There are words that express form. . . . And in every symbolization hitherto invented of mathematical logic there are symbols having constant formal meanings. . . . Such words or symbols express what are called 'logical constants.' Logical constants may be defined exactly as we defined forms; in fact, they are in essence the same thing. . . . In this sense all the 'constants' that occur in pure mathematics are logical constants.[3]

The difference between the new conception and the "old" logicism regarding mathematical constants is a matter of perspective. Both approaches are based on the equation that being mathematical = being formal = being logical. But while the classical logicists say that mathematical constants are essentially logical, the new conception implies that logical constants are esentially mathematical. Thus if the classical thesis is "the logicist thesis of mathematics," the new one is "the mathematical thesis of logic." Another point of difference worth noting is that according to the new conception, mathematical constants are logical only when construed as higher-level. Accordingly, the natural numbers, as individuals, are not logical objects. But as second-level entities, classes of classes, they are. This view is, as we saw in chapter 2, in some respects very Fregean. Frege's *logical* definition of the natural numbers takes numbers to be higher-level entities, i.e., classes of classes or classes of concepts. Indeed, the formulation of numerical statements as first-order quantifications in UL is exactly the same as Frege's in *The Foundations of Arithmetic*.

3 Mathematics and Logic

My discussion of logicism above highlighted one aspect of the relationship between logic and mathematics: in the new conception of logic any mathematical constant can play the role of a logical term, subject to certain requirements on its syntactic and semantic definitions. However, mathematical constants appear in the new logic also as extralogical constants, and this reflects another side of the relationship between logic and mathematics: as logical terms, mathematical constants are constituents of logical frameworks in which theories of various kinds are formulated and their logical consequences are drawn. But the "pool" of formal terms that can figure as logical constants is created in mathematics. The semantic definition of, say, the logical quantifier "there are uncountably many x" is based on some mathematical theory of sets. Similarly, the semantic definition of the quantifier "there is an odd number of x" is based on arithmetic. And even in standard logic, the semantic definitions of the truth-functional connectives and the universal (existential) quantifier are based on certain simple Boolean algebras. These observations point to a difference between logic and mathematics vis-à-vis formal terms: formal terms are *created* in mathematics; they are *used* in logic.

Now since logic provides a framework for theories in general, the meaning of formal terms can be given by a mathematical theory formulated within logic. We can thus picture the interplay between logic and mathematics as a cumulative process of definition and application. Starting with a logical system that applies certain elementary but powerful mathematical functions (Boolean truth functions, the universal/existential-quantifier function and, usually, identity) to a first-level extralogical vocabulary, we construct various formal theories. Such theories describe mathematical structures by delimiting the semantic variability of the extralogical terms of the language. This is done by introducing a set of extralogical axioms that partition the "universe" of all models for the language into those that do, and those that do not, "realize" the theory. In this way the axioms of the theory give specific meanings to all nonlogical terms of the language. Once mathematical terms are defined within the framework of standard first-order logic, they can be incorporated in the superstructure of a new, extended system of logic. As an example, consider the first-order theory of Peano arithmetic. As soon as arithmetic terms receive their meaning within this theory, we can convert them into logical arithmetic quantifiers: the numerical quantifiers, the "even" quantifier, quantitative comparative quantifiers ("there are fewer x's such that ... than x's such that____"), and so on. We can now use the new logical vocabulary to formulate theories—mathematical, physical, etc.—that assume the existence of a machinery for counting and comparing sizes. In these theories we will *logically* conclude that, say, there are 4 *B*s, given that there are 2 *C*s, and that the number of *B*s is twice the number of *C*s. As we shall see below, there is an essential difference between applying mathematics by using mathematical terms as part of the logical superstructure and applying mathematics by adding extralogical mathematical constants and axioms to a theory of standard first-order logic.

4 Ontological Commitments of Theories

Quine is known for the thesis that the logical structure of theories in a standard first-order formalization reflects their ontological commitments. To determine the ontology of a theory $\mathscr{T}$ formulated in natural language (or a scientific "dialect" thereof), we formalize it as a (standard) first-order theory, $\mathscr{T}_1$, and examine those models of $\mathscr{T}_1$ in which the extralogical terms receive their intended meaning. $\mathscr{T}$ is committed to the existence of such objects as populate the universes of the intended model(s) of $\mathscr{T}_1$. Thus if $\mathscr{T}$ includes a sentence of the form

(1) Uncountably many things have the property P,

then, since the notion of uncountably many is not definable in pure standard first-order logic, we have to include in $\mathscr{T}_1$ some theory in which "uncountably many" can be defined. Choosing a set theory with *Ur*-elements, we express (1) as

(2) $(\exists x)[x$ is a set & x is uncountable &
$(\forall y)(y \in x \rightarrow y$ is an individual & $Py)]$.

And through (2), $\mathscr{T}$ is committed to the existence of sets.

Now, consider what happens if we formalize $\mathscr{T}$ within the framework of UL, using a system $\mathscr{L}$ that contains, in addition to the standard logical terms and axioms, the logical quantifier "uncountably many" and appropriate axioms (e.g., Keisler's). Obviously, we do not need set theory to express (1) in $\mathscr{L}$. The meaning of (1) is adequately captured by the sentence

(3) (Uncountably many $x)Px$,

which does not commit $\mathscr{T}$ to the existence of sets. So with a "right" choice of logical vocabulary, $\mathscr{T}$ can be formalized by a theory, $\mathscr{T}_2$, whose ontology consists merely of individuals, not sets.

We see that the new conception of logic allows us to save on ontology by augmenting the logical machinery. We can weaken the ontological commitments of theories by parsing more terms as logical. We no longer talk about *the* ontological commitment of an unformalized (or preformalized) theory $\mathscr{T}$ (there is no such thing!). Instead, ontological considerations become a factor in choosing logical frameworks for formalizing theories.

The examination of Quine's principle from the perspective of UL reveals the relativistic nature of his criterion. The comparison of $\mathscr{T}_1$ and $\mathscr{T}_2$ highlights the crucial role played by logical constants in deciding commitment in other theories of logic and ontology as well. Consider the simple, straightforward view that the commitment of a theory under a formalization $\mathscr{F}$ is determined by what is common to all models of $\mathscr{F}$. Here too the difference in logical terms between the formalizations $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{T}$ results in essentially different commitments. The occurrence of the quantifier "uncountably many" in (3) ensures that in every model of

$\mathscr{T}_2$, P is assigned an uncountable set of individuals. But by the Löwenheim-Skolem theorem, $\mathscr{T}_1$ has at least one model in which the predicate "x is uncountable" is given a nonstandard interpretation and P is assigned a countable set. We can thus say that $\mathscr{T}_2$ is committed to an ontology of uncountably many objects, whereas $\mathscr{T}_1$ is not.

We see that logical terms are vehicles of *strong ontological commitment*, while extralogical terms transmit a relatively weak commitment. This difference in ontological import between logical and extralogical terms is explained by the fact that logical terms are semantically pre-fixed, whereas the meaning of extralogical terms is relative to models. To use Putnam's turn of speech, extralogical terms are viewed from *within* models, whereas logical terms are viewed from the *outside*.[4] Including formal terms as part of the logical superstructure allows us to use them in the logic with "a view from the outside."

The distinction between "strong" and "weak" ontological commitments explains the difference between using mathematical notions as part of the logical machinery and using them as extralogical terms in theories within the logic. It also suggests a guideline for choosing logical frameworks. If you formulate, say, a physical theory and you want to use formal tools created elsewhere (i.e., in some mathematical theory), you might as well include the mathematical apparatus as part of the logical superstructure. This will reflect the fact that you are not interested in specifying the meanings of the mathematical terms but in saying something about the physical world, using mathematical notions which you take as given. The pre-fixed notions will enable you to make some very strong claims about the physical world, strong in the sense that what they say does not vary from one model of the theory to another. All this will be done without compromising the usefulness of the logical framework in determining necessary and formal consequences. If, on the other hand, your goal is to define the mathematical notions themselves, you cannot construe them as logical, because as such their meaning would have to be given at the outset. You have to use undefined terms of the language (i.e., extralogical terms) and then construct a theory that will give these notions a distinctive content.[5]

5 Metaphysics and Logic

What role, if any, does metaphysics play in logics based on Tarski's ideas? First, for Tarski, the very notion of semantics has a strong metaphysical

connotation. Semantics investigates concepts having to do with the relationship between language and the world (see page 39). The categories used in classifying relevant features of the world are, ipso facto, an important factor in the analysis of such concepts. More specifically, as we have seen earlier in the book, it is crucial for Tarski that an adequate system of logic yield consequences that hold *necessarily* of reality. In that way metaphysics provides an important criterion for evaluating logical systems vis-à-vis their goal. But the role of metaphysics does not end with this external criterion. To see the metaphysical dimension of Tarskian semantics more clearly, it might be well to contrast his model-theoretic method with another type of theory, which, following Etchemendy 1990, I will call "interpretational." The interesting feature of *interpretational semantics* from my point of view is that it purports to ensure the satisfaction of Tarski's metaphysical condition by purely *syntactic* means. The interpretational definition of "logical consequence" is the following:

DEFINITION LC″ The sentence X is a *logical consequence* of the set of sentences K iff there is no permissible substitution for the nonlogical terms in the sentences of K and in X that makes all the former true and the latter false.

(A substitution is permissible if it is uniform and it preserves syntactic categories.) This definition, in essence, goes back to Bolzano (1837). It can also be found in modern texts, e.g., Quine's *Philosophy of Logic* (1970).

The distinctive feature of the interpretational test for logical consequence is that it is based on substitution of strings of symbols. Definition (LC″) does not take into account anything but grammar and the distribution of truth values to all the sentences of the language. Thus to the extent that syntactic analysis and a list of truth values are all that are needed to determine logical truths and consequences, interpretational semantics has nothing to do with metaphysics.

Tarski rejected the substitutional definition of "logical consequence" just for that reason. The success of interpretational semantics depends on the expressive power of the language. Relevant possible states of affairs may not be taken into account if the language is too poor to describe them. Thus, consider a language in which the only primitive nonlogical terms are the individual constants "Sartre" and "Camus" and the predicates "x is active in the French Resistance" and "x is a novelist." In this language the sentence

(4) Sartre was active in the French Resistance

will come out logically true under the substitutional test. But obviously, (4) is not necessarily true.

Etchemendy pointed out another problem with interpretational theory due to its syntactic character.[6] In interpretational "semantics," as in model-theoretic semantics, "logical consequence" and the other logical concepts are defined relative to a set of logical constants. But in interpretational semantics, the set of logical constants is an *arbitrary* set of terms, arbitrary because the interpretational theory does not offer a guide for determining whether a term is logical or not. Logical and extralogical terms are defined by use, and for all that interpretational semantics has to say, any term might be used either way. What Quine calls the remarkable concurrence of the substitutional and model-theoretic definitions of "logical consequence" for standard first-order logic is no more than a "happy" accident.[7] Since the standard logical constants do not form a grammatically distinct group, they are, from the point of view of interpretational semantics, indistinguishable from other terms that can also be held constant in the substitutional test. Thus even if every individual, property, and relation "participating" in relevant possible states of affairs has a name in the language, some divisions of terms into the logical and extralogical will yield unacceptable results. Suppose, for instance, that expressions naming Sartre and the property of being active in the French Resistance are included in the set of *fixed* (i.e., logical) terms. Then (4) will again turn out to be logically true. (See chapter 3.)

Tarski's semantics avoids the two problems indicated above by using a semantic apparatus which allows us to represent the relationship between language and the world in a way that distinguishes formal and necessary features of reality. The main semantic tool is the *model*, whose role is to represent possible states of affairs relative to a given language. Since *any* set of objects together with an "interpretation" of the nonlogical terms within the set determine a model, every possible state of affairs vis-à-vis the extralogical vocabulary is represented (extensionally). Furthermore, the choice of logical constants is constrained by the requirement that the logical superstructure represent formal, metaphysically unchanging parameters of possible states of affairs. (It should be noted that "possibility" in this context is "formal possibility." Therefore, the totality of models reflects "possibilities" that in general metaphysics might be ruled out by nonformal considerations. That is to say, the notion of possibility underlying the choice of models is wider than in metaphysics proper.)

Although metaphysical considerations are central to Tarskian semantics, only the most basic and general metaphysical principles are taken into account. The historical Tarski expressed a dislike for "abstruse" philosophical theories. The notions of necessity and possibility he used were, he emphasized, the common, everyday notions, not the philosopher's. I think Tarski's mistrust of philosophy is not warranted, but the claim that the philosophical foundation of logic should not rest on the web of philosophical controversies regarding modalities appears to me sound. Thus the view underlying the new conception of logic, that the mathematical "coordinates" of reality do not change from one possible world to another (and therefore mathematical constants can, in general, play the role of logical constants), is based on a basic, generally accepted belief about the nature of reality.

We cannot rule out, however, divergence of opinions even with respect to "core" metaphysical principles. And for those who do not share the "common" belief regarding the nature of mathematical properties, I propose the following relativistic view of logic: we can look at the definition of "logical terms" in chapter 3 as a schema saying that to treat a term as logical is to take it as naming a rigid, formal property or function (fixed across possible states of affairs) and define it in accordance with conditions (C) to (E). It is then left for the user to determine whether or not it is appropriate to treat a given term in that way. (A similar strategy will enable one to reconcile nominalistic compunctions with the new conception: depending on the metalinguistic resources one finds acceptable, one will construe those mathematical predicates that are definable in one's language as logical constants.)

The foundations of Tarskian semantics reach deep into metaphysics, but the link between models and reality may have some weak joints. In particular, Tarski has never shown that the set-theoretic structures that make up models constitute adequate representations of all (formally) possible states of affairs. This issue is beyond the scope of the present book, but two questions that may arise are the following: Is it formally necessary that reality consist of discrete, countable objects of the kind that can be represented by *Ur*-elements (or other constituents) of a standard set theory? Does the standard model-theoretic description of all possible states of affairs have enough parameters to represent all relevant aspects of possible situations (relevant, that is, for the identification of formally necessary consequences)? These and similar questions lie at the bottom of nonstandard models for physics, probablistic logic, and, if we put aside formality, such discourse theories as "situation semantics."

6 Proof-Theoretic Perspective

The philosophical justification of the new conception of logic is based on an analysis of certain *semantic* principles underlying modern logic. What about proof theory? Should we not set proof-theoretic standards for an adequate system of logic, for example, that it be *complete* relative to an "acceptable" deductive apparatus? The new logic, one would then object, surely fails to comply with this requirement! I think this judgement is premature. The "new conception of logic" is a result of reexamining the philosophical ideas behind logical semantics in response to certain mathematical generalizations of standard semantic notions (Mostowski and others). There is no sense in comparing the generalized semantics with current un- or pre-generalized proof theory. To do justice to the new conception from a proof-theoretic perspective, one has to cast a new, critical look at the standard notion of proof. This task may be exacting because there is no body of mathematical generalizations in proof theory directly parallel to "generalized logic" in contemporary model theory. However, if the new philosophical extension of logic based on semantics is significant, it poses a challenge to proof theory that cannot be overlooked. I can put it this way: if Tarski is right about the basic intuitions underlying our conception of logical truth and consequence, and if my analysis is correct, namely that these intuitions are not exhausted by standard first-order semantics, then since standard first-order logic has equal semantic and proof-theoretic power (completeness), these intuitions are not exhausted by standard first-order proof theory either. Semantically, we have seen, it suffices to enrich the superstructure of first-order logic by adding new logical terms. But what has to be done proof-theoretically? I hope that future researchers will take up this question as a challenge.

Appendix

Chapter 2,[1] Section 2

DEFINITION 1 Let A be a set. A *quantifier on A* is a function

$q : P(A) \rightarrow \{T, F\}$

such that if $m : A \rightarrow A$ is an automorphism (permutation) of A, i.e., m is one-to-one and onto A, then for every $B \subseteq A$,

$q(m(B)) = q(B)$,

where $m(B)$ is the image of B under m.

It is easy to see that Boolean combinations of quantifiers on A are also quantifiers on A.

DEFINITION 2 Let α be a cardinal number. A *2-partition of α* is a pair of cardinals (β, γ) such that $\beta + \gamma = \alpha$.

DEFINITION 3 Let $(\beta, \gamma)_\alpha$ be the class of 2-partitions of α. A *cardinality function on 2-partitions of α* is a function $t : (\beta, \gamma)_\alpha \rightarrow \{T, F\}$.

THEOREM 1 (Mostowski 1957.) Let A be a set. Let $\mathscr{T}$ be the set of cardinality functions on 2-partitions of $\alpha = |A|$. Let $\mathscr{Q}$ be the set of quantifiers on A. Then there exists a one-to-one function h from $\mathscr{T}$ onto $\mathscr{Q}$ defined as follows:

For any $t \in \mathscr{T}$, $h(t) =$ the quantifier q on A such that for any $B \subseteq A$,

$q(B) = t(|B|, |A - B|)$.

I will symbolize a quantifier q on A as Q_A. Given a quantifier on A, Q_A, I will call the cardinality function t satisfying the above equation the *cardinality counterpart* of Q_A and symbolize it as t_A^Q.