5.2
LOGICAL QUANTIFIERS

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This chapter offers a logical, linguistic, and philosophical account of modern quantification theory. Contrasting the standard approach to quantifiers (according to which logical quantifiers are defined by enumeration) with the generalized approach (according to which quantifiers are defined systematically), the chapter begins with a brief history of standard quantifier theory and identifies some of its logical, linguistic, and philosophical strengths and weaknesses. It then proceeds to a brief history of generalized quantifier theory and explains how it overcomes the weaknesses of the standard theory. One of the main philosophical advantages of the generalized theory is its philosophically informative criterion of logicality. The chapter describes the work done so far in this theory, highlights some of its central logical results, offers an overview of its main linguistic contributions, and discusses its philosophical significance.

I Standard Modern Quantifier Theory

1 Historical Beginnings

Modern quantifiers were first introduced by Frege (1879). The quantifiers All and Some were already recognized as logical operators by Frege's predecessors, going all the way back to Aristotle. But Frege offered a new analysis of these operators that enabled him to deal with multiple-quantifier prefixes and, more generally, multiple nesting of quantifiers. This, in turn, enabled him to extend traditional logic from a logic of "terms" (1-place predicates standing for properties) to a logic of relations. These changes revolutionized logic. Among the many achievements of Frege's theory, three are especially relevant for the present discussion:

(a) Frege's theory generated a considerably larger array of quantifiers than traditional logic: Starting with All as his basic logical quantifier, Frege construed not just the traditional Some, No, and Not All as (defined) logical quantifiers but also infinitely many others, for example, Exactly *n*, At Least *n*, and At Most *n*, for every natural number *n*.

(b) Frege's theory offered a systematic interpretation of multiple-quantifier prefixes and other nesting quantifiers.

(c) Frege's expansion of logical quantifiers gave rise to languages with enormous expressive power. Extensive parts of natural language as well as the entire language of classical mathematics and many segments of the language of science are expressible using his quantifiers.

Frege regarded 1st-order quantifiers as 2nd-order functions or concepts. The 1st-order quantifier Some is the 2nd-order concept of nonemptiness, the 1st-order All is the 2nd-order concept of universality. Using contemporary terminology, "$(\exists x)\Phi x$" says that the extension of Φ is not empty, and "$(\forall x)\Phi x$" says that the extension of Φ is universal.

Frege sanctioned both 1st-order and higher-order quantifiers. The former quantify over individual variables, i.e., variables ranging over individuals; the latter quantify over functional and predicative variables, i.e., variables ranging over functions, properties, and relations. For reasons that go beyond the scope of the present chapter (but see discussion of completeness in Section II.1 below), most logicians, linguists, and philosophers today focus on 1st-order quantifiers. In what follows I will limit myself to such quantifiers.

Note: Throughout this chapter I am using contemporary terminology.

2 Semantic and Proof-Theoretic Definitions of Quantifiers

Tarski (1933) developed a semantic definition of truth for languages formulated within the framework of modern logic. The recursive entries for the standard quantifiers—All ($\forall$) and Some ($\exists$)—are: given an assignment g of objects (in the intended universe) to the variables of the language,

($\exists$) "$(\exists x)\Phi x$" is true under g iff (if and only if) at least one object *a* (in the universe) satisfies "Φx" under g;

($\forall$) "$(\forall x)\Phi x$" is true under g iff every object *a* (in the universe) satisfies "Φx" under g;

where *a* satisfies "Φx" under g iff it satisfies "Φx" when all the free variables of "Φx" other than x are assigned values by g.

Gentzen (1935) developed a proof-theoretic method for defining logical constants in terms of *introduction* and *elimination rules*. Such rules tell us, for a given logical constant, under what conditions we can derive a statement governed by it from a statement (or statements) not governed by it, and under what conditions we can derive a statement (or statements) not governed by it from one governed by it. The entries for the standard logical quantifiers are, essentially:

($\exists$) *Introduction:* $\Phi(a) \vdash (\exists x)\, \Phi x$
Elimination: $(\exists x)\, \Phi x \vdash \Phi(a)$ provided "a" plays a merely auxiliary role

($\forall$) *Introduction:* $\Phi(a) \vdash (\forall x)\, \Phi x$ provided "a" is arbitrary
Elimination: $(\forall x)\, \Phi x \vdash \Phi(a)$

where "a" is an individual constant and certain syntactic conditions are given for " 'a' plays a merely auxiliary role" and " 'a' is arbitrary." Frege's, Tarski's, and Gentzen's characterizations identify roughly the same quantifiers, though the last generalizes differently from the first two.

II Strengths and Weaknesses of Standard Quantifier Theory

1 Logical Results

The logic of the standard quantifiers, i.e., standard 1st-order logic, is rich in logical, or rather meta-logical, results. Three results that are especially relevant for our discussion are:

(a) *The Löwenheim-Skolem Theorem* (1915/20): If a sentence σ (set of sentences Σ) has a model, it has a countable model, i.e., a model whose cardinality (i.e., the cardinality of its universe) is smaller than or equal to the cardinality of the natural numbers ($\aleph_0$).

(By definition, σ/Σ has a model iff there is a model M such that σ/all the sentences of Σ is/are true in M.) This result is naturally viewed as a limitation on the expressive power of standard 1st-order logic: Within the framework of this logic we cannot "seriously" define the notion of "uncountably many" and many other infinitistic notions, since every consistent statement of the form "There are uncountably many F's" comes out true in some countable model, i.e., a model in which there are only countably many F's. This result is sometimes called "Skolem's Paradox," although it is not a real paradox. But it does imply that the representational powers of the standard 1st-order quantificational languages are limited, since they cannot determine uniquely the intended meanings of formal terms and, as a result, the meaning of claims (or theories) containing such terms. For example, a consistent theory that says there are uncountably many objects has a model (i.e., is true in a model) that does *not* have uncountably many objects.

(b) *The Completeness Theorem* (Gödel 1930): Standard 1st-order logic has a complete proof system. I.e., A sentence σ follows logically from a set of sentences Σ in the semantic sense ($\Sigma \models \sigma$) iff it follows logically from it in the proof-theoretic sense ($\Sigma \vdash \sigma$).

The completeness result is perhaps the most important theoretical result of standard quantification theory. It says that in standard 1st-order logic the proof-theoretic relation of logical consequence is just as powerful as the semantic relation of logical consequence. This result is one of the main reasons most contemporary philosophers and logicians prefer standard 1st-order logic to standard (full) 2nd-order logic. (Full) 2nd-order logic is incomplete: Its proof system is less powerful than its semantic system.

One consequence of the completeness of standard 1st-order logic is the co-extensionality of the semantic and proof-theoretic definitions of the standard quantifiers. Since proofs, unlike models, are finite structures, this might be viewed as an advantage of the proof-theoretic definition over the semantic definition.

(c) *The Compactness Theorem* (Gödel 1930). Two significant formulations are: (i) If σ follows logically from an infinite set Σ, then it follows logically from a finite subset of Σ; (ii) If every finite subset of Σ has a model, Σ has a model.

The compactness result is naturally viewed as an advantage: it enables us to deal with infinite sets by dealing with their finite subsets.

It should be noted, however, that the connection of these theorems to the standard 1st-order quantifiers is not always simple. For example, compactness and completeness fail for logics like $L_{\infty,\omega}$, which has formulas with arbitrarily large conjunctions and disjunctions, but finitely many quantifiers.

2 *Linguistic Strengths and Weaknesses*

Standard quantifier theory made a substantial contribution to linguistic semantics and syntax through the works of Geach (1962), Montague (1973), Kamp (1981), May (1985), Partee (1987), and numerous others. Not only did it allow linguists to analyze complex linguistic structures involving multiple-quantifier prefixes (e.g., "$(\forall x)(\exists y)$," "$(\exists x)(\forall y)(\exists z)$") and other nesting quantifier structures (e.g.," $(\forall x)[(\exists y) \ldots \supset (\exists y)$—]"), it enabled them to formulate important questions concerning quantifier scope, indefinite nouns, anaphora, etc. A well-known problem of this kind is the quantificational structure of "donkey sentences," i.e., sentences like

(1) Every farmer who owns a donkey, beats it.

(See discussion in, e.g., Neale 1990.) Moreover, the theory of standard quantifiers played a central role in the development of important linguistic theories (for example, LF theory), the establishment of new semantic frameworks (for example, Montague grammar), and so on.

But the expressive power of the standard quantifiers is linguistically very limited. Even such simple quantitative determiners as "most", "few", "half", "an even number of", "all but ten", "finitely many", "uncountably many", and so on, cannot be expressed by any (well-formed) combination of standard 1st-order quantifiers. As far as quantifier-structures in natural language are concerned, it is quite clear that standard quantifier theory offers no more than a preliminary framework for their study.

3 *Philosophical Strengths and Weaknesses*

Standard quantifier theory has had a privileged status in contemporary analytic philosophy since its birth in the early years of the 20th century. It is interwoven into most analytic theories, and its position has been sealed by such works as Quine's influential *Philosophy of Logic* (1970). Even philosophers who have challenged the standard theory have rarely viewed its choice of quantifiers as a problem. They have assigned $\forall$ and $\exists$ a large array of alternative interpretations, going all the way from the substitutional interpretation (Marcus 1972) to the intuitionistic (Dummett 1973) and paraconsistent (Priest & Routley 1989) interpretations, but few have contested its restriction to the standard quantifiers. Perhaps the most celebrated feature of the standard theory is the *completeness* of its proof method, i.e., the fact that it is integrated into a powerful logic with a complete proof system (see above). This, indeed, is a desirable feature. For many, including Quine, the incompleteness of stronger quantifier theories justifies their rejection.

But standard quantifier theory is also widely recognized to have many problems. Here I will focus on six problems of philosophical significance (some mentioned above, and some interrelated):

(a) *Inability to Account for Basic Inferences*

In his 1962 abstract, Rescher points out that standard "quantificational logic is unequal to certain childishly simple valid arguments" like "Most things are A's, Most things are B's; therefore: Some A's are B's", and "Most C's are A's, Most C's are B's; therefore: Some A's are B's" (ibid.: 374). Since these inferences are of the same kind as the inferences logic is designed to account for (for example," All things are A's, Some things are B's; therefore: Some A's are B's"), this state of affairs is problematic.

(b) *Limited Logical Expressive Power*

As noted in Section II.2 above, many elementary quantificational notions, including quantitative determiner notions like "most", "few", "half", "finitely many", and "uncountably many" are indefinable in terms of the standard quantifiers (and other standard logical constants).

(c) *Limited Formal Expressive Power*

As demonstrated by the Löwenheim-Skolem Theorem (Section II.1(a) above), many formal (mathematical) notions, and in particular infinitistic notions like "uncountable", are not only indefinable as logical notions within the standard quantificational framework but are also inadequately definable as nonlogical notions within that framework. That is, it appears that the standard framework does not enable us to define them in a way that adequately captures their meaning. (In logicians' terminology, it enables us to define "uncountable" only as a "relative" notion but not as an "absolute" notion.)

(d) *Absence of a Theoretical Criterion for Logical Quantifiers*

It is a surprising fact about modern logic that it has a theoretical, precise, systematic, informative, and philosophically explanatory criterion for logical *connectives* but not for logical *quantifiers* or predicates. Logical connectives (at least in classical logic) have a precise and philosophically informative criterion, *truth-functionality*, which is given a mathematically precise definition by Boolean algebra. Every logical connective is represented by a Boolean operator in a 2-element Boolean algebra, and every Boolean operator of that kind determines a logical connective. Philosophically, every truth-functional connective—i.e., a connective that takes into account only the formal pattern of truth-values of its arguments—is logical, and every logical connective is truth-functional. (On the philosophical ramifications of this criterion, see (e) below.)

In contrast, for logical quantifiers and predicates we have nothing more than an utterly uninformative, unexplanatory, and unsystematic characterization, namely, a characterization by *enumeration*. All this characterization says is that $\forall$ and $\exists$ are logical quantifiers and $=$ is a logical predicate, and that any quantifier definable from $\forall$ and $\exists$ (using logical predicates and connectives) is also a logical quantifier, and any predicate defined from $=$ (using other logical constants) is also a logical predicate. Why *these rather than others* are logical, why *only these* are logical, remains a mystery. We may try to justify the standard choice on pragmatic grounds, by appealing to custom or convenient features of the associated logic (for example, completeness), but unless we add a theoretical dimension to such a justification it remains philosophically shallow. Furthermore, even on purely pragmatic grounds, how do we know that a stronger 1st-order system, i.e., a 1st-order system with additional logical quantifiers, will not give us an overall more desirable logic? The current situation is thus philosophically unacceptable.

(e) *Lack of Philosophical Understanding of Logicality*

Given the central role that logic plays in all areas of knowledge, including philosophy, the nature of logicality requires a theoretical account. The nature of logicality, however, has so far resisted such an account. One of the problems facing us is finding a fruitful standpoint from which to launch the investigation, i.e., a standpoint, or a perspective, which will enable us to formulate the question in a precise, systematic, and philosophically enlightening terms. It is quite clear that such a perspective requires an informative account of the logical constants. We know that logical truth, logical consequence, logical consistency, etc., are largely due to the logical constants involved; understanding the nature of the logical constants is therefore a prerequisite to a theoretical understanding of logicality.

Our experience with sentential logic shows that a theoretical criterion for logical constants can, indeed, make a significant contribution to our philosophical understanding of logicality. Philosophically, the truth-functionality of the logical connectives means that they *abstract* from everything but the bare *truth* pattern of their arguments, and this explains why logical consequences based on these connectives preserve *truth* rather than other features of sentences. Furthermore, this, together with the fact that truth-functional connectives correspond to mathematical operators, i.e., operators governed by *necessary* laws, and the fact that the test for logical truth/consequence based on such connectives is truth/preservation-of-truth under *all possible* variations in truth patterns, explain some of the philosophically significant features of sentential logic. In particular, they explain its great generality, topic neutrality, immunity to (at least most) empirical discoveries, and necessity. A philosophically informative criterion for *logical quantifiers* (and *predicates*) is needed to extend this understanding to all of logic.

(f) *The Shaky Status of the Semantic Definition of Logical Consequence for Quantificational Logic.*

In his 1936 paper, "On the Concept of Logical Consequence," Tarski constructed a semantic definition of logical consequence for quantificational logic that established logical semantics (model theory) as one of the two major fields of contemporary logic, along with proof theory. Tarski's semantic definition says that:

(LC) $\Sigma \models \sigma$ iff every model of Σ is a model of σ (or: there is no model in which all the sentences of Σ are true and σ is false).

Tarski anchored his semantic definition in the intuitive idea that *a logical consequence is a necessary-and-formal consequence*. To determine the adequacy of his definition, he asked whether it accurately captured this idea. Investigating this question, he realized that the answer varies according to the choice of logical constants. For some choices of logical constants—for example, treating Is-a-Property-of-Napoleon as a logical quantifier or treating $\forall$ as a nonlogical constant—the answer is clearly negative, while for others, including the standard choice of logical constants, no conflict seems to occur.

But even in the latter case, Tarski realized, it is not clear that his definition fully captures the idea it was supposed to capture. Adding such nonstandard logical quantifiers as Finitely Many also accords with this idea. The scope and justification of the contemporary definition of logical consequence, thus, require a systematic criterion for logical quantifiers.

A solution to all the above problems is offered by *generalized quantifier theory*.

III Generalized Quantifier Theory

1 Brief History

The history of generalized quantifiers starts with Mostowski (1957). Mostowski suggested that we might generalize the standard notion of logical quantifier along two dimensions, syntactic and semantic. Syntactically, a logical quantifier is a variable-binding operator that generates new formulas from old formulas. Semantically, a logical quantifier over a universe A is a cardinality function from subsets of A to a truth-value, satisfying a certain invariance condition. For example, the existential and universal quantifiers over A, $\exists_A$ and $\forall_A$, are functions that, given a subset B of A:

(2) $\exists_A(B) = T$ iff the cardinality of B, $|B|$, is larger than 0; i.e., iff $|B| > 0$.
(3) $\forall_A(B) = T$ iff the cardinality of the complement of B in A is 0; i.e., iff $|A–B| = 0$.

It is easy to see that both $\exists$ and $\forall$ satisfy the following invariance condition:

(4) A quantifier Q on A is invariant under all permutations of A.

That is: For every A, B, B', where A is a nonempty set and B, B' are subsets of A: If B' is obtained from B by some permutation of A, then: $Q_A(B) = Q_A(B')$, where a permutation of A is a bijection p: $A \rightarrow A$, i.e., a 1–1 function from A onto A. Using "$\subseteq$" for "is a subset of" and "p^*" for "the function on subsets of A induced by p", we can spell out (4) by:

(5) If A is a nonempty set, $B \subseteq A$, $B' \subseteq A$, and there is a permutation p of A such that $p^*(B) = B'$, then: $Q_A(B) = Q_A(B')$.

Generalizing, Mostowski arrived at the following criterion for logical quantifiers:

(PER) A Quantifier Q is logical iff for any non-empty set A, Q_A is invariant under all permutations of A.

Under this criterion, every 2nd-order 1-place cardinality predicate is a 1st-order logical quantifier, and every 1st-order logical quantifier is a 2nd-order 1-place cardinality predicate. Examples of logical quantifiers in addition to $\exists$ and $\forall$, are Most (interpreted as, say, "more than half"), An Even Number of, Finitely Many, Uncountably Many, and α-Many for every cardinal number α, finite or infinite. Such quantifiers appear in English sentences such as

(6) Most things are beautiful.
(7) There is a finite number of stars.

They are symbolized by the formulas

(8) (Mx)Bx
(9) (Fx)Bx,

using obvious abbreviations.

Examples of 2nd-order 1-place predicates that are not logical quantifiers are "some men", "all women", "is a property of Napoleon", and "is a color property".

Lindström (1966) generalized Mostowski's criterion further in two ways. First, he extended it to 1st-order predicates of all types (for example, "is red" and "="), and second, he extended it to 1st-order quantifiers (2nd-order predicates) of all types (for example, "is a relation between humans" and "is symmetric").

Lindström's criterion for logical quantifiers and predicates can be formulated as follows:

(ISO) A Quantifier/predicate Q is logical iff for any Q-structure S, Q is invariant under all isomorphisms of S.

By a Q-structure S we mean a structure $<A, \beta_1, \ldots, \beta_n>$, where A is a nonempty set (the universe of S), and $\beta_1, \ldots, \beta_n$ are elements or structures of elements of A of the same types as the corresponding arguments of Q. Let $S = <A, \beta_1, \ldots, \beta_n>$ and $S' = <A', \beta'_1, \ldots, \beta'_n>$. We say that structures S and S' are isomorphic—$S \cong S'$—iff there is a bijection (a 1–1 and onto function) f from A to A' such that for $1 \leq i \leq n$, $f^*(\beta_i) = \beta'_i$, where f^* is induced by f. Q is invariant under all isomorphisms of Q-structures S iff for any Q-structures S, S', if $S \cong S'$, then $Q_A(\beta_1, \ldots, \beta_n) = Q_{A'}(\beta'_1, \ldots, \beta'_n)$.

Among the new quantifiers/predicates satisfying ISO are, in addition to all the above examples of quantifiers satisfying PER, the identity predicate of standard 1st-order logic as well as nonstandard quantifiers such as the 2-place Most and Only, and the 1-place Symmetric and Well-Ordered. As the last two examples suggest, logical quantifiers, according to ISO, are not limited to (although they include all) cardinality quantifiers. Examples of a predicate and a quantifier which do not satisfy ISO are "is red" and "is a relation between humans".

We can indicate the type of a given quantifier by an n-tuple representing the types of its arguments, using <1> for a property on, or a subset of, the universe; <1,1> for a pair of subsets of the universe; <2> for a 2-place relation on the universe, and so on. Quantifiers of types involving only 1's we call "predicative", those of types involving at least one number larger than 1 we call "relational" or "polyadic". Thus, the standard $\forall$ and $\exists$ are predicative quantifiers of type <1>, but the 2-place Most is a predicative quantifier of type <1,1>, and the 1-place relational (polyadic) Symmetric is of type <2>. The last two are defined, semantically, as follows:

Given a universe A, subsets B,C of A, and a binary relation R on A,
($M^{1,1}$) $(M^{1,1})_A(B,C) = T$ iff $|B \cap C| > |B - C|$.
(S^2) $(S^2)_A(R) = T$ iff R is symmetric.

They appear in English sentences such as:

(10) Most presidents are women.
(11) Equality is a symmetric relation.

Syntactically, these sentences have the logical forms:

(12) $(M^{1,1}x)(Px,Wx)$.
(13) (S^2xy) Exy.

Literally, (13) say, "Symmetric xy, x = y". It means something like "For x and y to be equal is for them to be symmetrically equal."

Tarski (1966) formulated essentially the same criterion, albeit in terms of permutations. As demonstrated by McGee (1996), a formulation in terms of isomorphisms is preferable.

Note: the presentation of this section is based on Sher (1991).

2 *Logical Results*

Generalized quantifier theory has had a considerable impact on the development of mathematical logic since the 1960s. It has led to the development of a new field, model-theoretic logic, and the establishment of numerous results. (For a representative collection of articles in this field see Barwise & Feferman 1985.) Two results of special interest are due to Lindström (1969, 1974) and Keisler (1970).

Lindström's Theorem

By offering a model-theoretic generalization of the notion of logical constant, Mostowski and Lindström created tools for comparing the strength of different 1st-order logical systems (with respect to their ability to define model-theoretic structures). These tools enabled Lindström to arrive at an important characterization of standard 1st-order logic:

(FOL) Standard 1st-order logic is the strongest logic that has both the compactness/completeness and the Löwenheim-Skolem properties (see Section II.1 above).

Keisler's Result

Keisler showed that completeness is not limited to standard 1st-order logic; some stronger logics are also complete. In particular 1st-order logic with the generalized quantifier Uncountably Many has a complete proof system. (In contrast, 1st-order logic with the quantifier At Least Denumerably Many does not.) Keisler's result weakens the case for the standard 1st-order quantifiers based on completeness (see II.1. and II.3. above).

3 *Linguistic Results*

Generalized quantifier theory has led to a renaissance in linguistic semantics. We can divide its linguistic contributions into two: contributions to the study of natural-language *determiners*, and contributions to the study of natural-language *polyadic (relational)* *quantifiers*. In both cases the restriction to *logical* quantifiers is relaxed, but here we will limit ourselves to logical quantifiers.

(a) Determiners

Determiners form an important linguistic category of noun modifiers (different from adjectives). Determiners include *quantity* modifiers that are naturally construed as quantifiers. Some of these—for example, "all", "every", "each", "any", "a", "some", "at least (exactly, at most) one"—are naturally construed as the standard quantifiers, $\forall$ and $\exists$, or as quantifiers defined in terms of these. However, many quantitative determiners cannot be analyzed using the standard quantifiers. A few examples are "most", "few", "half", "more . . . than—", "finitely many", "uncountably many", "$\aleph_0$", and so on. Generalized quantifier theory enables linguists to account for the semantics of all these determiners by treating them as predicative quantifiers. Among the pioneers in this field are Barwise & Cooper (1981), Higginbotham & May (1981), van Benthem (1983), Westerståhl (1985), and Keenan & Stavi (1986).

Quantifiers representing determiners are for the most part of type <1,1>, and sometimes of more complex types, such as <<1,1>,1>. We have seen above how $Most^{1,1}$ is defined. $\exists^{1,1}$ is defined as follows:

$$(\exists^{1,1})\quad \exists^{1,1}{}_A(B,C) = T \text{ iff } |B\cap C| > 0$$

We may say that $\exists^{1,1}$ is generated from $\exists^1$ by relativizing it to a subset of the universe, B. Most existential quantifications in English have this (or some derivative) form. In the simplest case, linguists assign

(14) Some B's are C's

the logical form

$$(15)\quad (\exists^{1,1}x)(Bx, Cx)$$

rather than

$$(16)\quad (\exists^1 x)(Bx \,\&\, Cx).$$

A determiner-quantifier of type <<1,1>1> appears in

(17) More students than teachers attended the lecture,

whose logical form is:

$$(18)\quad (O^{<1,1>,1}x)(Sx, Tx, Lx).$$

$O^{<1,1>,1}$ is defined by:

$$(19)\quad O^{<1,1>,1}{}_A(B,C,D) = T \text{ iff } |B\cap D| > |C\cap D|.$$

Linguists and logicians have developed a rich theory of the characteristic features of determiner-quantifiers. For an up-to-date account see Peters & Westerståhl (2006), Chs. 3–6.

(b) *Relational or Polyadic Quantifiers*

Generalized quantifier theory enables us to go beyond determiners to more complex quantificational structures. The study of relational or polyadic quantifiers in natural language has been pursued by Higginbotham & May (1981), Keenan (1987, 1992), van Benthem (1989), Sher (1991), and many others. The simplest relational quantifiers are of type <2>. We have already defined one relational quantifier of this type, Symmetric.

One way to arrive at relational quantifiers is by combining predicative quantifiers in some order. In the simplest case, relational quantifiers of this kind correspond to *linear* quantifier-prefixes (of other quantifiers); in more complex cases they correspond to non-linear, *branching* or *partially-ordered* quantifier-prefixes.

An example of a relational quantifier equivalent to a linear prefix of predicative quantifiers is $\forall\exists^2$, where

(20) $(\forall\exists^2 xy)\ \Phi xy$

is logically equivalent to

(21) $(\forall^1 x)(\exists^1 y)\ \Phi xy$.

While $\forall\exists^2$ simply says "for every' x there is a y such that", a more intricate relational quantifier that resembles it but cannot be expressed by a linear prefix of $\forall$'s and $\exists$'s, says "Every x has its own y". This quantifiers appears, for example, in

(22) For every drop of rain that falls, a flower grows,
(23) Every family has its own misery. (Every family is unhappy in its own way. Tolstoy)

If we symbolize this quantifier by "$\forall\rightarrow\exists^2$", the logical structure of (22) and (23) is:

(24) $(\forall\rightarrow\exists^2 xy)(Bx, Cy)$.

Other interesting relational quantifiers appear in:

(25) Different students answered different questions on the exam. (Keenan 1987)
(26) No three students answered the same number of questions. (ibid.)
(27) Most pairs of people are not married to each other.

An up-to-date discussion of relational (polyadic) quantifiers appears in Peters & Westerståhl (2006), Ch. 10.

A special type of relational quantifier corresponds to a *branching* quantifier-prefix. But the branching prefix is more interesting when thought of as a complex prefix. Two examples of irreducibly branching quantifier-prefixes are:

$$(28)\quad \begin{matrix}(\forall x)(\exists y) \\ (\forall z)(\exists w)\end{matrix}\ \Phi xyzw. \quad \text{(Henkin 1959)}$$

$$(29)\quad \begin{matrix}(M^1x) \\ (M^1y)\end{matrix}\ \Phi xy. \quad \text{(Barwise 1979)}$$

English sentences exemplifying these forms, or in the case of (29), the more complex form:

$$(30)\quad \begin{matrix}(M^{1,1}x)\psi_1 x \\ (M^{1,1}y)\psi_2 y\end{matrix}\ \Phi xy \quad \text{(Barwise 1979)}$$

are:

(31) Some relative of each villager and some relative of each townsman hate each other. (Hintikka 1973)
(32) Most boys in my class and most girls in your class have all dated each other. (Variation on Barwise 1979)

The branching form raises interesting questions of interpretation. In particular, there is no consensus on the interpretation of non-monotone-increasing branching quantifiers (Q^1 is monotone increasing iff $Q^1(B) = T, B \subseteq C \Rightarrow Q^1(C) = T$) and on how to formulate a general semantic definition of branching quantifications that will deal with branching prefixes of any complexity. Most importantly, the branching form raises the question of *compositionality*, i.e., the question of whether the interpretation of logically complex formulas is always determined by that of their simpler components. The key issue is the structure of the branching prefix, which need not be a tree structure (but only a partial ordering). In addition to the works mentioned above, see Westerståhl (1987) and Sher (1990, 1997). For a further development (IF languages) and a discussion of compositionality see Hintikka & Sandu (1989, 2001).

4 Philosophical Results

Generalized quantifier theory provides solutions to all the philosophical problems plaguing the standard theory that were discussed in Section II.3 above. How successful these solutions are, is an open question. But generalized quantifier theory undoubtedly makes an important contribution to philosophy by creating a framework for confronting philosophical questions that were earlier thought to resist any systematic investigation. This is especially true of the last three questions, (d)–(f), which concern fundamental issues in the philosophy of logic.

Briefly, the solutions to the six problems raised in Section II.3 are:

(a) Basic (Non-standard) Logical Inferences

The problem of recognizing and accounting for basic inferences based on quantifier-notions is largely, and perhaps fully, solved by the generalized theory. All inferences based on such quantitative notions as "most", "few", "half", "finitely many", "as many as", etc., can be accounted by this logic.

(b) *Logical Expressive Power*

This problem is also largely (perhaps fully) solved by generalized quantifier theory. *All* common quantitative notions, including all quantitative determiner notions, can be construed as logical notions in this theory. Indeed, every classical mathematical notion can be construed as a logical quantifier, predicate, or function according to this theory, often by transforming it to a 2nd-order notion. As a result, the expressive power of generalized 1st-order logic as a whole is very large. (See McGee 1966.)

(c) *Formal Expressive Power & the Löenheim-Skolem Theorem*

In many systems of generalized 1st-order logic the Löenheim-Skolem theorem does not hold. As a result, formal or mathematical notions that lose their intended meaning in standard 1st-order logic retain their intended meaning in such systems. There are two possible ways of achieving this result: by constructing such notions as logical, and by constructing them as nonlogical within an appropriate generalized logical system. The first was noted in (b) above. We can, for example, construct the formal notion "uncountably many" as a logical quantifier, UNC^1, defined, for an arbitrary universe A by:

$$(UNC^1)\quad UNC^1_A(B) = T \text{ iff } |B| > \aleph_0.$$

Since the interpretation of logical constants is built into the apparatus of models (see Sher 1991), UNC^1 will preserve its intended meaning in every model. Sometimes, however, we prefer to define a formal notion as a nonlogical notion (for example, in order to explain its meaning or establish its properties). In this case we might be able to construct an appropriate generalized framework that blocks the Löenheim-Skolem theorem, and define the desired concept within that framework without subjecting it to the Löenheim-Skolem effect. (The gain in expressive power is not totally free; often, the price includes loss of compactness and other properties that make standard 1st-order logic easier to work with.)

(d) *A Theoretical Criterion for Logical Quantifiers*

Generalized quantifier theory provides a criterion (necessary and sufficient condition) for logical quantifiers and predicates—ISO—that is as theoretical, systematic, precise, and comprehensive as is the truth-functional, Boolean criterion for logical connectives. Both criteria are essentially invariance criteria. The truth-functional criterion says that a sentential connective is logical iff it does not distinguish anything about its arguments besides their bare truth value. ISO says that a quantifier/predicate is logical iff it does not distinguish anything about its arguments besides their bare formal structure. That is, logical connectives are invariant under all truth-preserving variations in their arguments; logical quantifiers are invariant under all structure-preserving variations in their arguments. Both criteria are also precise mathematical criteria: The sentential criterion is based on Boolean algebra, the quantificational criterion is based on set theory (most commonly ZFC). Finally, both criteria determine a maximal collection of logical constants, connectives, and quantifiers, respectively. In so doing, both sanction the existence of a far larger number of logical constants than one would anticipate based on (mere) intuition, indeed an infinite number of logical connectives and quantifiers. (The fact that we cannot reduce the collection of logical quantifiers to a small finite collection as in the case of the logical connectives is to be expected. The power of quantificational logic is so much greater than that of sentential logic that it cannot have as simple a logical apparatus as the latter's.)

(e) Philosophical Understanding of Logicality

For reasons indicated in the discussion of this topic in Section II.3 above, a theoretical criterion for logical constants can be instrumental in providing a philosophically informative characterization of logicality. What is the philosophical content of ISO? The idea of invariance under isomorphisms is a systematic rendering of the intuitive idea of being indifferent to, or not distinguishing, the individual characteristics of objects in any given universe of discourse. There is a very long tradition of viewing this idea as characterizing logic. Two examples, taken from Kant and Frege, are:

> [General logic] treats of understanding without any regard to difference in the objects to which the understanding may be directed.
>
> (Kant 1981/7: A52/B76)

> Pure logic . . . disregard[s] the particular characteristics of objects.
>
> (Frege 1879: 5)

ISO offers a precise characterization of this condition and places it at the center of our understanding of logic. To be logical is to abstract from, or disregard, everything about objects and their relations besides their formal features or structure. That is, to be logical is to be formal or structural in a very strong sense. (It is, in a certain sense, to be maximally formal or structural.)

This characterization explains some of the intuitive features of logic: its high degree of necessity, extreme generality, topic neutrality, strong normativity, and even apriority (or quasi-apriority). Very briefly, since logical constants are formal, the laws governing them are necessary. Since logical constants apply regardless of what objects exist in the domain (universe of discourse), their laws are highly general and topic neutral. Since they are not affected by the empirical features of objects (or the laws governing these features), their laws are (largely) nonempirical. And since their invariance is greater than that of physical and other constants (which are not invariant under *all* isomorphisms), the laws associated with them are normatively stronger than those associated with other constants (logic does not have to obey the laws of physics, but physics has to obey the laws of logic). For sources and further discussion see Tarski (1966), Sher (1991, 1996, 2008), McGee (1996), and others.

(f) Foundation of Logical Consequence

By providing a general criterion for logical constants (combining ISO for predicates and quantifiers and the Boolean criterion for logical connectives), generalized quantifier theory enables us to evaluate the adequacy of Tarski's semantic definition of logical consequence. The question is whether Tarski's model-theoretic test yields intuitively necessary and formal consequence in languages with logical constants satisfying the above criterion. The answer appears to be positive. Since (i) Tarskian consequences depend on the logical constants of the given language and on the apparatus of models; since (ii) according to the above criterion, logical constants are formal (in the sense of taking into account only the formal or structural features of their arguments); and since (iii) Tarskian models represent all formally possible situations relative to the given language, consequences based on formal constants satisfying Tarski's test are necessary and formal.

For a view claiming that the problem of logical constants is a "red herring", see Etchemendy (1990). For a discussion of this and the other issues raised in (d)–(f), see Sher (1991, 1996, 2008).

5 *Criticisms and Alternatives*

To date, generalized quantifier theory has been criticized mostly on two grounds.

Some of the criticisms (for example, Hanson 1997 and Gómez-Torrente 2002) concern the applicability of ISO to natural language quantifiers. The critics present examples of natural-language quantifier expressions that purportedly satisfy ISO yet are intuitively nonlogical. They conclude that the characterization of logical constants should be pragmatic rather than theoretical.

Other criticisms (for example, Feferman 1999) focus on more theoretical issues, like the relation between logic and mathematics and certain meta-mathematical features of quantifiers satisfying ISO (although here too, "gut feeling" plays a central role). Two logicians (Feferman 1999 and Bonnay 2008) offer alternatives to ISO, for example, invariance under homomorphisms (Feferman). For response to some of these criticisms see Sher (2001, 2003, 2008).

Related Topics

1.3 Logical Form
1.8 Compositionality
3.4 Quantifiers and Determiners
3.6 Anaphora
4.4 The Role of Mathematical Methods
5.1 Model Theory
5.9 Montague Grammar
7.4 Frege, Russell, and Wittgenstein

References

Barwise, J. (1979) "On Branching Quantifiers in English," *Journal of Philosophical Logic* 8: 47–80.
Barwise J. & R. Cooper (1981) "Generalized Quantifiers and Natural Language," *Linguistics & Philosophy* 4: 159–219.
Barwise, J. & S. Feferman (eds.) (1985) *Model-Theoretic Logics*, New York: Springer-Verlag.
Bonnay, D. (2008) "Logicality & Invariance," *Bulletin of Symbolic Logic* 14: 29–68.
Dummett, M. (1973) "The Philosophical Significance of Intuitionistic Logic," in *Truth and Other Enigma*, Cambridge, MA: Harvard University Press, 1978, pp. 215–47.

Etchemendy, J. (1990) *The Concept of Logical Consequence*, Cambridge, MA: Harvard University Press.
Feferman, S. (1999) "Logic, Logics, and Logicism," *Notre-Dame Journal of Formal Logic* 40: 31–54.
Frege, G. (1879) "Begriffsschrift, a Formula Language, Modeled upon That of Arithmetic, for Pure Thought," in J.v. Heijenoort (ed.) *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Cambridge, MA: Harvard University Press, 1967, pp. 1–82.
Gärdenfors, P (ed.) (1987) *Generalized Quantifiers: Linguistic & Logical Approaches*, Dordrecht: D. Reidel.
Geach, P.T. (1962) *Reference & Generality: An Examination of Some Medieval & Modern Theories*, Ithaca: Cornell University Press.
Gentzen, G. (1935) "Investigations into Logical Deduction," in M.E. Szabo (ed.) *The Collected Papers of Gerhard Gentzen*, Amsterdam: North-Holland, 1969, pp. 68–131.
Gödel, K. 1930. "The Completeness of the Axioms of the Functional Calculus of Logic", in *Collected Works*, Vol. I, Feferman et al. (eds.), New York: Oxford University Press, 1986, pp. 103–123.
Gómez-Torrente, M. (2002) "The Problem of Logical Constants," *The Bulletin of Symbolic Logic* 8: 1–37.
Hanson, W.H. (1997) "The Concept of Logical Consequence," *Philosophical Review* 106: 365–409.
Henkin, L. (1959) "Some Remarks on Infinitely Long Formulas", in *Finitistic Methods: Proceedings of the Symposium on Foundations of Mathematics*, Warsaw: P.W.N. and Pergamon Press, 1961, pp. 167–83.
Higginbotham, J. & R. May (1981) "Questions, Quantifiers and Crossing," *Linguistic Review* 1: 41–79.
Hintikka, J. (1973) "Quantifiers vs. Quantification Theory," *Dialectica* 27: 329–58.
Hintikka, J. & G. Sandu (1989) "Informational Independence as a Semantic Phenomenon," in J.E. Fenstad et al. (eds.), *Logic, Methodology, and Philosophy of Science*, Amsterdam: Elsevier, pp. 571–89.
—— (2001) "Aspects of Compositionality," *Journal of Logic, Language, and Information* 10: 49–61.
Kamp, H. (1981) "A Theory of Truth & Semantic Representation,"in J. Groenendijk et al. (eds.), *Truth, Interpretation, and Information*, Dordrecht: Foris, 1984, pp. 1–41.
Kant, I. (1981/7) *Critique of Pure Reason*, Tr. N. Kemp Smith, London: Macmillan, 1929.
Keenan, E.L. (1987) "Unreducible *n*-ary Quantifiers in Natural Language," in Gärdenfors 1987: 109–50.
—— (1992) "Beyond the Frege Boundary," *Linguistics and Philosophy* 15: 199–221.
Keenan, E.L. & J. Stavi (1986) "A Semantic Characterization of Natural Language Determiners," *Linguistics & Philosophy* 9: 253–329.
Keisler, H.J. (1970) "Logic with the Quantifier 'There Exist Uncountably Many'," *Annals of Mathematical Logic* 1: 1–93.
Lindström, P. (1966) "First Order Predicate Logic with Generalized Quantifiers," *Theoria* 32: 186–95.
—— (1969) "On Extensions of Elementary Logic," *Theoria* 35: 1–11.
—— (1974) "On Characterizing Elementary Logic," in S. Stenlund (ed.), *Logical Theory and Semantic Analysis*, Dordrecht: D. Reidel, pp. 129–46.
Marcus, R.B. (1972) "Quantification and Ontology," *Noûs* 6: 240–50.
May, R. (1985) *Logical Form: Its Structure and Derivation*, Cambridge, MA: MIT Press.
McGee, V. (1996) "Logical Operations," *Journal of Philosophical Logic* 25: 567–80.
Montague, R. (1973) "The Proper Treatment of Quantification in Ordinary English," in *Formal Philosophy: Selected Papers*, R.H. Thomason (ed.), New Haven: Yale University Press.
Mostowski, A. (1957) "On a Generalization of Quantifiers," *Fundamenta Mathematicae* 44: 12–36.
Neale, S. (1990) *Descriptions*, Cambridge, MA: MIT Press.
Partee, B.H. (1987) "Noun Phrase Interpretation and Type-Shifting Principles," in J. Groenendijk et al. (eds.), *Studies in Discourse Representation Theory and the Theory of Generalized Quantifiers*, Dordrecht: Foris, pp. 115–143.
Peters, S. & D. Westerståhl (2006) *Quantifiers in Languag eand Logic*, Oxford: Oxford University Press.
Priest, G. & R. Routley (1989) "The Philosophical Significance and Inevitability of Paraconsistency," in G. Priest et al. (eds.) *Paraconsistent Logic Essays on the Inconsistent*, Munich: Philosophia Verlag, pp. 483–539.
Quine, W.V. (1970) *Philosophy of Logic*, 2nd edition, Cambridge, MA: Harvard University Press 1986.
Rescher, N. (1962) "Plurality-Quantification," Abstract, *Journal of Symbolic Logic* 27: 373–4.
Sher, G. (1990) "Ways of Branching Quantifiers," *Linguistics and Philosophy* 13: 393–422.
—— (1991) *The Bounds of Logic: A Generalized Viewpoint*, Cambridge, MA: MIT Press.
—— (1996) "Did Tarski Commit 'Tarski's Fallacy'?," *Journal of Symbolic* Logic 61: 653–86.
—— (1997) "Partially-Ordered (Branching) Generalized Quantifiers: A General Definition," *Journal of Philosophical Logic* 26: 1–43.
—— (2001) "The Formal-Structural View of Logical Consequence," *Philosophical Review* 110: 241–61.
—— (2003) "A Characterization of Logical Constants *Is* Possible," *Theoria* 18: 189–97.

—— (2008) "Tarski's Thesis," in D. Patterson (ed.), *New Essays on Tarski and Philosophy*, Oxford: Oxford University Press, pp. 300–39.
Tarski, A. (1933) "The Concept of Truth in Formalized Languages". In J. Corcoran (ed.) *Logic, Semantics, Metamathematics*, 2nd ed. Indianapolis: Hackett, 1983, pp. 152–278.
—— (1966) "What Are Logical Notions?" *History and Philosophy of Logic 7* (1986): 143–54.
Van Benthem, J. (1983) "Determiners and Logic," *Linguistics and Philosophy* 6: 447–78.
—— (1989) "Polyadic Quantifiers", *Linguistic sand Philosophy* 12: 437–64.
Westerståhl, D. (1985) "Logical Constants in Quantifier Languages," *Linguistics and Philosophy* 8: 387–413.
—— (1987) "Branching Generalized Quantifiers and Natural Languages," in Gärdenfors, pp. 269–98.