Chapter 2

The Combinatory

1. Although Leibniz generally supported and adopted the logic of the Schools, while at the same time correcting and completing it, it would be a mistake to believe that his own logic was only a development or refinement of that of Aristotle. Leibniz himself suggests the opposite, when, after having bestowed on Aristotle through the mouth of Theophilus, the sumptuous praise we have read above (Chap. 1, §1), he has Philalethes deliver the following compliment: "You appear to defend common logic, but I see clearly that what you are presenting belongs to a much higher logic, which stands to the common sort as erudition does to the letters of the alphabet."[1] It is to this higher logic and not to that of Aristotle that he gives the title "universal mathematics."[2] And it is this that we must now investigate and explain.

However novel Leibniz's logic was to be, the root idea was nonetheless suggested to him by Aristotle, as he reports in several passages.[3] He was not yet twelve years old when he immersed himself with pleasure in the intricacies of scholastic logic: he delighted in the books of Zabarella, Rubio, and Fonseca, and he read Suarez as easily as Milesian fables or novels. Already eager for innovation and tormented by the desire to invent, he cast onto paper criticisms and projects of reform; he acknowledged later that he experienced great pleasure in rereading the drafts he had written when he was fourteen years old. At that age, he had an idea that was to be the germ of his entire logic. He had noticed that the predicables or categories of Aristotle serve to classify simple terms (or concepts) in the order in which they give content to propositions. He wondered, and inquired of his teacher, why complex terms (that is, propositions) should not be classified in the same way, in the order in which they give content to syllogisms or, in general, deductions. He admits that he did not know, and his teachers undoubtedly also did not know, that this is precisely what geometers do when they arrange their theorems in the order in which they are deduced. Thus, even before he had become acquainted with it, the mathematical method already constituted his ideal logic. It is therefore not surprising that he later took it as a model and guide, and that he conceived of logic as a universal mathematics.

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[1] *New Essays*, IV.xvii.7.

[2] Philalethes: "I am beginning to form an entirely different idea of logic from that which I had before. I took it to be a game for schoolboys, but now I see that, in your conception of it, it is like a universal mathematics." *New Essays*, IV.xvii.8.

[3] *The Life of Leibniz Briefly Sketched by Himself* (Guhrauer, II, Notes, 52; Klopp, I, xxxvii); *Foundations and Specimens of the General Science of William Pacidius* (*Phil*., VII, 126; see also ibid., 127: "There fell to me the simple, as I was still but a child, yet fertile beginnings of a certain great art."); *Phil*., VII, 185, 292. Cf. *Elements of Reason* (LH IV 7B, 6 Bl. 7 recto): "While still a boy, possessing only the rudiments of common logic and ignorant of mathematics, the plan arose in me—by what inspiration I do not know—that an analysis of concepts could be devised from which truths could be extracted through certain combinations and evaluated like numbers. It is pleasant even now to recall by what arguments, however youthful, I came to the idea of so great a thing."

2. His teachers, unable to satisfy his questions, were content to respond that a child should not attempt to introduce innovations; but in vain they preached docility and respect for tradition to this autodidact, who came to know the sciences only by reinventing them.[4] Thus he continued to meditate on his idea of a classification of judgments. In this way he was led to imagine that all truths can be deduced from a small number of simple truths by an analysis of the concepts they contain, and that from these in turn, all ideas can be reduced by decomposition to a small number of primitive, undefinable ideas. Thereafter, it would suffice to make a complete enumeration of these simple ideas, the genuine elements of all thought, and to combine them, in order to obtain progressively and by means of an infallible procedure all the complex ideas. In this way, one would construct the *alphabet of human thoughts*, and all derivative notions would be no more than combinations of the fundamental notions, just as the words and sentences of speech are no more than combinations, indefinitely varied, of the 26 letters of the alphabet.[5]

3. It was in his eighteenth year that Leibniz seems to have conceived this project,[6] whose idea caused him to rejoice, he says, "assuredly because of youthful delight,"[7] though he did not yet comprehend its full significance. But his further meditations only confirmed him in his plan by exposing its merit and fruitfulness. In addition, he soon made part of it public in *On the Art of Combinations*, in which he showed that one of the principal applications of the art of combinations is logic, and more particularly, the logic of invention.[8] Indeed, from what has already been said, we can see that whatever makes use of the science of combinations must be the art of invention, since knowing how to combine the simple concepts is all that is required in order to be able to find all the truths that express their relations, and as a result to discover new ones. This application to logic was undoubtedly one of the main factors that led Leibniz (who was not yet a mathematician[9]) to develop, if not to create in every aspect,[10] the new branch of mathematics today called combinatorics.[11]

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[4] *Phil*., VII, 185.

[5] "When I applied myself more intently to the matter, I inevitably came upon this wonderful idea, namely that a certain alphabet of human thoughts could be devised, and that by the combination of the letters of this alphabet, and by the analysis of the words formed from them, all things could be discovered and decided" (*Phil*., VII, 185). See also the letter to Tschirnhaus of 1679, quoted in Chap. 4, §5 (*Math*., IV, 482; *Brief*., IV, 405-6).

[6] Leibniz to Oldenburg, ca. 1676 (*Phil*., VII, 12). Cf. *On the Organon or Great Art of Thinking*: "The *alphabet of human thoughts* is a catalogue of those things that are conceived through themselves, and by their combination, all our other ideas are produced" (LH IV 7C Bl. 156 verso). "The *alphabet of human thoughts* is a catalogue of *primitive notions* or of those things we cannot render clearer by means of definitions" (LH IV 7C Bl. 160).

[7] *Phil*., VII, 185.

[8] Application X, no. 55 (*Phil*., IV, 61; *Math*., V, 39). Leibniz in fact distinguishes the analytic (that is, demonstrative logic, or the syllogisms of Aristotle) from inventive logic (i*bid*., no. 10). The very subtitle of *On the Art of Combinations* ("The Origins of the Logic of Invention") indicates that its principal application is the art of invention.

[9] *Phil*., IV, 12, note.; *Math*., III, 71. Cf. LH IV 7B, 6 Bl. 7 recto; Leibniz to Gabriel Wagner, 1696 (*Phil*., VII, 522).

[10] He had in fact been preceded by other inventors, notably Pascal, whose *Traité du Triangle Arithmétique*, composed before 1654 and printed in 1662 (after his death), was made public only in 1665. Leibniz, however, does not appear to have known of it (Cantor, II, 749; III, 38). Moreover, he himself

4. Leibniz expresses the fundamental problem of the logic of invention in the following manner: "Given a subject, find all its possible predicates; given a predicate, find all its possible subjects." In other words, find all the *true* propositions in which a concept appears, whether as a subject or as a predicate. As a proposition is a combination of two terms, a subject and a predicate, the problem is again one of combinations. Here Leibniz was inspired by the example of Ramon Lull (1234-1315), who in order to find all the possible propositions among nine terms arranged them in a circle and drew all the lines joining them in pairs (that is, the sides and diagonals of the nonagon thus formed).[12] This produced 36 lines, representing the same number of propositions; and, in fact, the number of combinations of nine objects taken two by two is 36, or (9 x 8) / (1 x 2).

The *Ars magna* of Ramon Lull, however, was a more general method that would allow the formation of every conceivable proposition. For this purpose, Lull had drawn up a table of categories, divided into six series, and had distinguished within these series nine absolute attributes, nine relations, nine questions, nine subjects, nine virtues, and nine vices.[13] In each series, the total number of combinations that can be formed from the nine terms taken one by one, two by two, etc., is $511 = 2^9-1$. As a result, if one combination from each series is combined with one combination from each of the others, an immense total number of combinations is obtained, for it is equal to 511^6 = 17,804,320,388,674,561.

This is the calculation Leibniz carries out by applying the laws of combinations. Ramon Lull, who did not know these laws, had imagined a mechanical procedure for realizing the combinations (somewhat at random), or at least some of them. He took six concentric circles of increasing radius, movable around their center, and inscribed the nine terms of each series on the corresponding circle. In order to obtain (in principle) all the possible combinations, it would suffice to turn each of the circles, one relative to the others, and by noting at each position the six terms located on the same radius, to combine them with each other.[14] Leibniz makes a subsequent allusion to this mechanical procedure[15] and mentions several other inventors who employed it to diverse ends. De Breissac applied it to military and strategic questions.[16] Harsdörffer,[17] from whom

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mentions the theorems he borrows from others, and cites their authors: Schwenter, Cardan, and Clavius (*Phil*., IV, 38; *Math*., V, 15-16).

[11] In German, *Combinatorik*. Unlike mathematicians, we do not call it *combinatorial analysis*, (1) because this science has nothing in common with what is customarily called "analysis," and (2) because, as we shall see later (Chap. 7, §7), this science is for Leibniz essentially synthetic and not analytic. Cf. *Foundations of a General Science* (Erdmann, 86a).

[12] See his *Cabala* and *Ars Magna*. Leibniz also cites Cornelius Agrippa, *Commentarium in artem brevem Lullii*, and J.H. Alsted, *Architectura Artis Lullianae* (found in his *Thesaurus artis memorativae*). Leibniz borrowed from Lull his title *On the Organon or Great Art of Thinking* (LH IV 7C Bl. 156-7).

[13] See Note II for this table.

[14] In fact, one can obtain in this way only the combinations of six terms each belonging to a different series, whose number is only $9^6 = 531,441$.

[15] "Ramon Lull also constructed a wheel." *On the Art of Combinations* , no. 96 (*Phil*., IV, 74; *Math*., V, 52). One finds in Leibniz's manuscripts a *Circle of Wisdom after Lull, Gregory Tolosanus*, etc. (LH IV 7C Bl. 19), as well as a sheet bearing an engraved figure of the circles of Lull, under the title, *The Great Art* (LH IV 7C Bl. 3). The *Ars magna sciendi* of Athanasius Kircher also contains some diagrams of movable concentric circles. See Note II.

[16] *On the Art of Combinations*, nos. 92, 93. Concerning this subject, Leibniz corrected an error of Harsdörffer: De Breissac employed nine circles of six terms each; the number of combinations of nine terms each belonging to a different circle is $6^9 = 10,077,696$ and not 216, as Harsdörffer believed. This

Leibniz borrows this curious information, made use of it in an attempt to form all the words of the German language. Finally, J.H. Alsted, author of an encyclopedia of which there are numerous traces in the works of Leibniz,[18] had employed the same procedure to draw up a table of topics in his *Thesaurus artis memorativae*. Leibniz compares these logical contrivances to secret locks (*pensiles serae*, *serae armillares*), composed of several movable circles or rings carrying certain marks or letters, whose security consists entirely in the huge number of combinations they admit, such that there is only a very small probability of finding the unique combination that opens them.

5. While appreciating the ingenuity of the principle, Leibniz is thoroughly critical of the *Ars magna* of Ramon Lull, as much because of the arbitrary choice of so-called general and simple concepts as because of their number, artificially fixed at nine in each series in order to obtain an appearance of symmetry. He even finds fault with the admission of certain series, like that of questions, which duplicates that of attributes, and those of virtues and vices, which obviously are not primitive and universal notions. In short, Lull's invention constitutes a topic useful to rhetoric rather than a table of categories appropriate to the needs of philosophy.[19]

Despite this unfavorable judgment, it was appropriate to recall the name of Ramon Lull, for his "great art" at first captivated Leibniz, by his own admission,[20]and as a result, he was to be no stranger to the idea of his "combinatorial characteristic." Among his precursors, Leibniz further cites Pierre Grégoire of Toulouse, author of *Syntaxis artis mirabilis*, and above all, the Jesuit priest Athanasius Kircher, who had just published his *Polygraphia nova et universalis ex combinatoria arte detecta* (Rome, 1663), of which we will have more to say later. Finally, Leibniz recalls in passing Hobbes's ingenious thought that all reasoning is calculation, from which Hobbes had not known how to draw any profit but which Leibniz was to develop and deepen in an original manner,[21] and he passes to the exposition of his own invention, which on the whole seems to owe little to his precursors. Let us see, then, how Leibniz conceives the application of the combinatory to the logic of invention.

6. To begin, one must analyze every concept by defining it, that is, by reducing it to a combination of simpler concepts. In this way, we arrive at a certain number of absolutely simple concepts, which are both irreducible and indefinable.These will be the first-order

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example shows the degree to which one is misled in evaluating the number of possible combinations when ignorant of the laws of mathematics.

[17] Georg-Philipp Harsdörffer of Nuremburg (1607-1658) had published two supplements (1651, 1652) to the 1636 *Deliciae physico-mathematicae* of Daniel Schwenter of Nuremburg (1585-1636).

[18] See Chap. 5, §5.

[19] One will recall the scornful judgment Descartes passed on the art of Lull, as well as on syllogistic logic, in his *Discourse on the Method* (Part II).

[20] In a letter to Remond of July 1714, after having spoken of a Count Jörger, "a great admirer of the universal art of the celebrated Ramon Lull," who "prefers Lull to all the moderns, even to M. Descartes," he adds: "When I was young, I took some pleasure from the art of Lull; but I believed that I had discerned in it a number of flaws, of which I have said something in a small schoolboy's essay, entitled *On the Art of Combinations*" (*Phil*., III, 619-20). Cf. Leibniz's *Judgment on the Writings of Comenius*, where he speaks of "the art of combinations as having been distorted and disgraced by the excesses of the Lullians" (Dutens, V, 181). See Note XIII.

[21] See Appendix II, "Leibniz and Hobbes."

terms, which will be organized in a class by themselves (the first), and designated by some commonplace signs (the simplest being numerals). In a second class will be organized terms of the second order, obtained by combining in pairs those of the first; then in a third class the terms of the third, obtained by combining in threes the terms of the first, and so on. Each term formed, being a combination of simple terms, will be represented by the (symbolic) product of the corresponding marks or numerals, which will constitute at the same time its definition. But in order to simplify the notation as well as the definitions, each term of the third order will be the product of one term of the first and one term of the second, each term of the fourth order a product of either two second-order terms or one first-order term and one third-order term, and so on. In order to designate terms of an order higher than the first, one would use a fractional symbol, in which the denominator indicates the number of the order and the numerator the number of the term in that class. Thus 1/2 would designate the first term of the second class, 2/3 the second term of the third class, and so on.[22]

It is easy to see that the same term will be expressible in several ways, according to how one combines the simple terms that enter into it as factors. In order to verify the equivalence of these different expressions, that is, the identity of the term expressed, it suffices to reduce them to simple terms and thereby to recover the primitive definition of the term in question.[23] This operation is, we see, analogous to the decomposition of numbers into prime factors, and it has the same advantage, namely that the logical decomposition of a complex term into simple terms can be carried out in only one way, such that that term has a unique and well-defined formula that supplies an infallible criterion of its identity.[24]

7. Granting this, it is easy to find all the (logically) possible predicates of a given subject. All the factors or divisors of a given term are its predicates, since they express each and every one of the characteristics or qualities that underlie part of its comprehension and define it. One can therefore attribute to a given subject, first, each of its prime factors, and then each of the combinations formed from these.

By proceeding in order, according to the general method of the art of combinations, one thus discovers all the possible predicates of a given subject, from its simple terms up to the subject itself; the proposition of which it is the subject and of which the predicate is the product of all its simple terms, can be considered its proper definition, that is, an identity.[25] The number of possible predicates is easy to calculate: if k is the number of

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[22] See Note VI for the example Leibniz gives of this method applied to geometry.

[23] Thus, following Leibniz's own example, if the four simple terms were 3, 6, 7, 9, their primitive combinations would be 3.6, 3.7, 3.9, 6.7, 6.9, 7.9. The same terms of the third class would be represented as 1/2.9, 3/2.6, 5/2.3, all of these being expressions equivalent to 3. 6. 9. Cf. *On Universal Synthesis and Analysis*, in which Leibniz recalls *On the Art of Combinations* (*Phil*., VII, 293). See Chap. III, §7; Chap. VI, §9.

[24] *On the Art of Combinations*, §§64-70 (*Phil*., IV, 65; *Math*., V, 42). To illustrate this property with an arithmetical analogy, the number 210 can occur as a product in different forms (42 x 5, 6 x 35, 14 x 15, etc.), but when one decomposes the factors into prime factors, they all reduce to a single expression, which is the decomposition of 210 into prime factors (2 x 3 x 5 x 7).

[25] This method can be illustrated by means of the same arithmetical analogy. Suppose we want to discover all the factors of 210. We first take its prime factors (2, 3, 5, 7), then their two-by-two combinations (2 x 3, 2 x 5, 2 x 7, 3 x 5, 3 x 7, 5 x 7), then their three-by-three combinations (2 x 3 x 5, 2 x 3 x 7, 2 x 5 x 7, 3 x 5 x 7), and finally their (unique) four-by-four combination (2 x 3 x 5 x 7), which is the

simple terms (prime factors) that enter into the definition (or formula) of a given term, there will be as many different predicates (or divisors) as there are total combinations of k letters, namely 2^k-1. If one has sorted the terms formed according to the method indicated above, the number k will be precisely the number of the class to which the given term belongs.[26]

8. Leibniz next considers the inverse of the preceding problem: given a term, find all its possible subjects. Since each term is a predicate of all the products in which it figures, the problem reduces to that of finding all the combinations that contain a given combination.[27] It is clear that if k is the number of prime factors of the given combination and n the *total* number of simple terms, the number of combinations needed is the number of possible combinations of (n - k) distinct terms, or 2^{n-k} - 1, but as the given term can be its own subject (in an identity statement or definition), and as this case corresponds to the combination of 0 terms, we can add 1 to the preceding number and obtain from it the total number 2^{n-k}.[28]

9. In order to find the number of particular predicates of a given term (that is the number of predicates that can be attributed to it in a particular proposition), Leibniz argues as follows: a particular affirmative proposition can be derived from a universal proposition

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number 210 itself. In *On Universal Synthesis and Analysis*, Leibniz states the same theory using letters. Consider, e.g., the concept $y = abcd$. It has as simple elements (prime factors) the concepts a, b, c, d. We form their two-by-two combinations, and let $ab = l$, $ac = m$, $ad = n$, $bc = p$, $bd = q$, $cd = r$. We form their three-by-three combinations, and let $abc = s$, $abd = v$, $acd = w$, $bcd = x$. All these combinations are among the predicates of y, but the *convertible* predicates are only *ax, bw, cv, ds; lr, mq,* and *np*, each of which is equivalent to *abcd*. Thus, "given any species, the propositions demonstrable of it, or its predicates—both those more general than it and those that are convertible—can be enumerated in order, and from these the most noteworthy can be selected" (*Phil*., VII, 293).

[26] *On the Art of Combinations*, no. 72 (*Phil*., IV, 66; *Math*., V, 44). Leibniz erred in this calculation. Let $k = 4$. There are four one-by-one combinations, six two-by-two combinations, four three-by-three combinations, and one four-by-four combination, for a total of 15 combinations. On the other hand, one term gives rise to one combination, two terms give rise to three (= 2^2 - 1) combinations, three terms give rise to seven (= 2^3 - 1) combinations, and four terms gives rise to 15 (= 2^4 - 1) combinations. Leibniz believes it is necessary to multiply each of the first numbers by each of the second, and to take the sum of the products; thus he obtains: (4 x 1) + (6 x 3) + (4 x 7) + (1 x 15) = 4 + 18 + 28 + 15 = 65. But it is easy to see that this calculation rests on a fallacious argument, for each of the combinations of one, two or three terms reappears, with all of its subcombinations, among the 15 combinations of 4 terms, so that in all there are no more than 15 genuinely different combinations, or in general, 2^k - 1. Leibniz probably wanted to take into account all the different forms of the same combination (as in *On Universal Synthesis and Analysis*), but then each combination of h terms gives rise to only 2^h - 1 combinations of a distinct form (as many as there are different combinations of h - 1 terms). Thus one combination of two terms has only one form, one combination of three terms has 2^2 - 1 = 3 forms, one combination of four terms has 2^3 - 1 =7 forms, and so on, with the result that for four terms, the total number of combinations having a distinct form is: (4 x 1) + (6 x 1) + (4 x 3) + (1 x 7) = 29, as one can verify for the number 210.

[27] Leibniz calls that given combination which is the invariable element (factor) common to all the desired combinations the *origin*.

[28] *On the Art of Combinations*, nos. 74-78 (*Phil*., IV, 67; *Math*., V, 45). As in the example chosen by Leibniz, let 5 be the total number of simple terms: a, b, c, d, e. If the given term is a *simple* combination, a, the number of possible subjects is $2^{5-1} = 2^4 = 16$; if the given term is a *double* combination, ab, the number of possible subjects is $2^{5-2} = 2^3 = 8$; and so on. (Because he wrongly counts as distinct subjects the permutations of the terms of the identical subject, Leibniz is mistaken here in adding to 7, the number of proper subjects, 2, instead of 1, as the number of identical subjects.)

either by subalternation or by partial conversion.[29] While in subalternation the predicate issues from the predicate, in conversion it issues from the subject. Therefore, the number of particular predicates of a given term is equal to the sum of its universal predicates and universal subjects. An analogous argument leads to the conclusion that the number of particular subjects is equal to that of particular predicates, that is, to that which has just been defined.[30] This is obvious, moreover, in virtue of the symmetry of the particular affirmative, which can be simply converted. As a result, every particular affirmative subject of the given term is also a particular affirmative predicate of it.

10. At this point Leibniz considers an analogous problem for negative propositions, namely, to discover the negative subjects and predicates of a given term, both universal and particular. But the solution he offers is very incomplete and even false. For he states the following rule for calculating the number of negative predicates of a given term A, that is, the number of terms which can be substituted for X in the judgement "No A is X." One first evaluates the total number of combinations that can be formed from *n* simple terms; one subtracts from it the number of universal affirmative predicates of the given term and that of its proper (non-identical) universal affirmative subjects; the remainder will be the desired number. Now the number of all the possible combinations of *n* terms is $2^n - 1$. The number of predicates of a given term, assumed to be formed from *k* simple factors, is $2^k - 1$; and the number of its proper subjects is $2^{n-k} - 1$. Therefore, the number of its negative predicates will be given by the formula:

$$(2^n - 1) - (2^k - 1) - (2^{n-k} - 1) = 2^n - 2^k - 2^{n-k} + 1$$
$$=(2^k - 1)(2^{n-k} - 1).$$

But one can discover the number of negative predicates of a term by another method, and far more simply, by noticing that these predicates are all the combinations of the *n* terms in which there occur none of the factors of the given term, that is, all the combinations of $(n - k)$ distinct simple terms, the number of which is $2^{n-k} - 1$. This formula differs from that of Leibniz by the factor $2^k - 1$, so the former is accurate only in the case in which $k = 1$.

Leibniz adds: "And likewise with regard to subjects." Now it is clear that the universal negative subjects of a term are identical to its universal negative predicates, in virtue of the symmetry of the universal negative which can be simply converted: If no A is X, no X is A. The number of negative subjects is therefore the same as that of negative predicates. As regards particular negative subjects and predicates, Leibniz gives too vague and too brief an account for it to be discussed.[31]

[29] See Appendix I.
[30] *On the Art of Combinations*, nos. 73, 79 (*Phil*., IV, 66; *Math*., V, 44).
[31] *On the Art of Combinations*, no. 79 (*Phil*., IV, 68; *Math*., V, 46). The simplest and most certain method for evaluating the number of particular subjects and predicates would consist in considering the particular affirmative as the negation of the universal negative, and the particular negative as the negation of the universal affirmative. Given a term composed of *k* simple terms, one is able to find its $2^k - 1$ universal affirmative predicates, that is, those of the $2^n - 1$ concepts that can be truthfully attributed to it. Consequently, it is false to attribute to it (universally) any one of the others, and as a result it has just that many particular negative predicates, namely $2^n - 2^k$. Likewise, one is able to find its $2^{n-k} - 1$ universal affirmative subjects and its $2^{n-k} - 1$ universal negative subjects (or predicates); the remaining concepts

11. Leibniz then searches for the number of arguments (syllogisms) by which one can demonstrate a given proposition (conclusion). Let S be the subject and P the predicate. Leaving aside the case in which these two terms belong to the same class and as a result must be identical (this could be verified by decomposing them into simple factors), we see that S can only be the subject of P when it belongs to a higher-order class and contains all the simple factors of P. In order to make this relation perspicuous, we could employ as many middle terms as there are terms which are both predicates of S and subjects of P. To evaluate this number, it suffices to consider the predicates of S as the set of possible terms, and to find how many of the possible subjects in this set (excluding S) the term P possesses. Let n be the number of simple factors of S, k that of the simple factors of P (all understood, by hypothesis, to be among those of S, so that $k < n$). The desired number is, as we know, $2^{n-k} - 1$, and since one must exclude from it S itself, $2^{n-k} - 2$.[32]

As for the universal negative proposition, this can be proved if the subject and predicate, when reduced to their simple terms, have no term in common.[33] This is easily understood: if the subject (*abc*) and the predicate (*ade*) had in common a single simple term (factor) *a*, one could affirm "Some S is P"; for the term *abcde* would designate *some* S, and it would have as its predicate *ade*. One could not, therefore, affirm "No S is P." In order to find all the middle terms that can serve to prove this proposition, Leibniz gives the following rule: find all the predicates of the subject and all the subjects of the predicate, add the numbers, and one will have the desired number of middle terms. In fact, the proposed conclusion can be proved either by denying of the predicate P each of the predicates of the subject S, or by denying of the subject S each of the subjects of the

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(among the 2^{n-1} possibilities) therefore will be, respectively, its particular negative subjects and its particular affirmative subjects (or predicates), and their number will be, in each case, $2^n - 2^{n-k}$. This formula differs from the one Leibniz found, $(2^k - 1) + (2^{n-k} - 1)$, which shows that the latter is mistaken. And indeed, he arrived at it by means of a fallacious argument: he assumed that all particular affirmatives derive from an universal affirmative by subalternation or conversion. But this is not the case, for there are some particular affirmatives of the form "Some *ab* is *ac*" that do not derive from a universal having the same terms. (One in fact can affirm neither "Every *ab* is *ac*" nor "Every *ac* is *ab*.") Therefore, the number determined by Leibniz must be too small, which is indeed the case when $k > 1$. In sum, a term composed of k factors has:

(1)	$2^k - 1$	universal affirmative predicates
(2)	$2^{n-k} - 1$	universal affimative subjects
	—	universal negative predicates
	—	universal negative subjects
(3)	$2^n - 2^k$	particular negative predicates
(4)	$2^n - 2^{n-k}$	particular negative subjects
	—	particular affirmative predicates
	—	particular affirmative subjects.

[32] In other words, the number of combinations of the n terms of S that contain the k terms of P is, including S and P, 2^{n-k}, and hence, $2^{n-k} - 2$, when S and P are excluded.

[33] Leibniz says: "If once reduced to simple terms they are not contained in each other." This condition is not sufficient, for let there be, by way of example, the terms *abc* and *ad*: neither of the two is contained in the other, but nevertheless one can assert "Some *abc* is *ad*," or "Some *ad* is *abc*" (namely, *abcd*). Moreover, the stated condition, being simply the negation of the condition of the universal affirmative, can only be the condition of its contradictory, the particular negative "Some S is not P." And indeed, it is true to assert "Some *abc* is not *ad*."

predicate P. One will then have as many syllogisms as there are predicates of S or subjects of P. This results from the rules of the syllogism, according to which a universal negative conclusion can only be deduced from two universal premises, one of which is affirmative, the other negative.[34] If the major premise is negative, one denies a predicate[35] of the predicate P or a subject[36] of the subject S; if the minor is negative, one denies a predicate[37] of the subject S or a subject[38] of the predicate P.[39]

12. In sum, despite the errors of detail which taint this "schoolboy's essay,"[40] one cannot overestimate the importance of *On the Art of Combinations* in the history of Leibniz's thought and in the formation of his logic. It is interesting to recall the judgement he later delivered on this inaugural work. Although recognizing its imperfections, which originated from the extreme youth of the author and his ignorance of mathematics, he took pleasure in remembering it as the prelude to and the anticipation of his subsequent inventions.[41] He himself indicated, with an obvious willingness, what he found to be good and sound in his first work, that is, what he had preserved from it: "several new admirable thoughts, from which the seeds of the art of invention are scattered,... and among the rest, that excellent idea of an analysis of human thoughts into something like an alphabet of primitive notions."[42] In an unpublished fragment, he declares, on the subject of the same work: "there are some things which smack of the young man and apprentice, but the foundation is good and I have since built upon it...."[43] What he built upon it was the whole of his logic—the great work whose plan occupied him throughout his entire life.

It is important, therefore, to retain from *On the Art of Combinations* the main ideas which, by Leibniz's own admission, were to serve as the basis for his subsequent investigations. First, all concepts must be resolvable into simple concepts by an analysis analogous to the decomposition of numbers into prime factors; conversely, they must all be obtained and composed by the progressive combination of these simple concepts. Next, the simple concepts or categories, which are the constitutive elements of all the others, are of a rather small number; but this does not prevent them from giving rise to the innumerable multitude of complex concepts, thanks to the marvelous fertility of the art of combinations. It will suffice, therefore, to assign to each of them a name or simple

[34] The corresponding moods are: *Celarent* (first figure); *Cesare*, *Camestres* (second figure); *Camenes* (fourth figure).

[35] In *Celarent*.

[36] In *Cesare*.

[37] In *Camestres*.

[38] In *Camenes*.

[39] *On the Art of Combinations*, nos. 80-82 (*Phil*., IV, 69; *Math*., V, 46). We have employed throughout the expression "to deny of." This is because the universal negative is symmetrical or reciprocal (simply convertible), such that it denies the subject of the predicate as well as the predicate of the subject.

[40] Leibniz to Remond, July 1714 (*Phil*., III, 620). See n. 20.

[41] *Phil*., VII, 186.

[42] *Phil*., IV, 103. Note published by Leibniz in the *Acta Eruditorum* in 1691, concerning the reissue of his *On the Art of Combinations* (Frankfurt-on-Main, 1690), which was done without his authorization or knowledge. Cf. *New Essays*, IV.iii.18.

[43] *Project and Attempts to Arrive at Some Certitude, So as to End a Good Portion of the Disputes and to Advance the Art of Invention*, ca. 1686 (LH IV 6, 12e).

sign, so as to form the *alphabet of human thoughts*,[44] on the basis of which all other notions can be expressed by combining the signs in the same manner as the corresponding concepts. Finally, logic, and more precisely the art of invention, depends entirely upon the combinatory,[45] which instructs by discovering in order all the possible combinations of the simple concepts (or their signs), and by determining with certainty their relations of inclusion or exclusion, that is, by discovering all the truths relative to some concept. From there is born the idea of a *spécieuse générale* or universal characteristic: a logical algebra that would replace concepts by combinations of signs, propositions by relations among these signs, and reasoning by a sort of calculus. It would provide a universal and infallible method for demonstrating propositions and discovering new ones from them. It would be, in other words, both an art of judgment and an art of invention.

[44] See Leibniz to Oldenburg, 27 August 1676 (*Phil*., VII, 11; *Brief*., I, 199); *Confession of Nature Against Atheists*, 1669 (*Phil*., IV, 103); *On the Universal Science* (*Phil*., VII, 199).

[45] In Leibniz's manuscripts there are two plans for a new *On the Art of Combinations*, the more developed of which dates from 1680 (LH XXXV 1, 27b, c). There we see how the idea of the combinatory was progressively extended in his mind; it is identified with the art of invention. (For more detail see Chap. VII, §7). In the older of the two one still finds somewhat youthful inquiries, for example, "On the size of the book containing all the hexameters which can be written. On the book in which all the truths comprehensible to man are soon to be written down," which recall those found at the end of *On the Art of Combinations* of 1666. To these investigations it is appropriate to add a curious, unpublished essay entitled "*On the Horizon of Human Learning*, or a meditation concerning the total number of possible truths and falsities which men, such as we know them, can state; and concerning the number of feasible books. In which we demonstrate that these numbers are finite, and that it is possible to write and easy to imagine a number that is much greater. In order to show the limits of the human mind and its capacity to know them" (LH IV 5 Bl. 9; see Bodemann, 83; cf. LH IV 8 Bl. 19, published in Bodemann, 114; and LH IV 8 Bl. 25, which is a preface to *Horizon*). It is a simple exercise in the combinatory, which takes as its point of departure the number of letters in the alphabet (as if the number of possible propositions could depend on the number of letters, which varies according to the people and the times). Leibniz had written to Fontanelle of it on 20 February 1701. This is all the more remarkable as elsewhere he supports the exact opposite, namely, that the concepts and first truths are infinite in number (LH IV 6, 12 Bl. 23).

