Chapter 9

The Geometrical Calculus

1. At the beginning of the piece Erdmann entitled *On the Universal Science or Philosophical Calculus*, Leibniz, in the course of summing up his views on the importance of a good characteristic, indicates that algebra is not the true characteristic for geometry, and alludes to a "more profound analysis" that belongs to geometry alone, samples of which he claims to possess.[1] What is this properly geometrical analysis, completely different from algebra? How can we represent geometrical facts directly, without the mediation of numbers? What, finally, are the samples of this new method that Leibniz has left us? The present chapter will attempt to answer these questions.[2]

An essay concerning this geometrical analysis is found attached to a letter to Huygens of 8 September 1679, which it accompanied. In this letter, Leibniz enumerates his various investigations of quadratures, the inverse method of tangents, the irrational roots of equations, and Diophantine arithmetical problems.[3] He boasts of having perfected algebra with his discoveries—the principal of which was the infinitesimal calculus.[4] He then adds: "But after all the progress I have made in these matters, I am no longer content with algebra, insofar as it gives neither the shortest nor the most elegant constructions in geometry. That is why... I think we still need another, properly geometrical linear analysis that will directly express for us *situation*, just as algebra expresses *magnitude*. I believe I have a method of doing this, and that we can represent figures and even

[1] "Progress in the art of rational discovery depends for the most part on the completeness of the characteristic art. The reason why demonstrations are commonly sought only in the case of numbers and lines, and things represented by them, is that, besides numbers, there are no *manageable characters corresponding to concepts*. This is also the reason why not even geometry has been treated analytically, except insofar as it is reduced to numbers by means of an algebraic analysis [*per analysin speciosam*], in which arbitrary numbers are designated by letters. There is, however, a *deeper geometrical analysis* that uses its own characters, by means of which many things can be presented more elegantly and more succinctly than through algebra. I have examples of this at hand" (*Phil.*, VII, 198). This piece is after 1686.

[2] That this geometrical analysis is completely different from the infinitesimal calculus, with which it might have been confused, will become clear in this chapter from all the texts in which Leibniz asserts that it does not rest on the idea of number and has nothing in common with algebra, that is, with analytic geometry, of which infinitesimal geometry is an extension. Furthermore, we see that in *Foundations and Examples of the General Science*, Leibniz already possessed his infinitesimal geometry but had not yet developed his geometrical calculus, a fact which allows us to date this text between 1675 and 1679: "The elements of transcendental geometry will be conveyed here, so that for the first time it can be said that all geometrical problems are within our power" (an allusion to the insufficiency of Descartes's analytic geometry; see Chap. 7, §4). "Nevertheless, the author does not promise to give a method here for always finding the best constructions, for this requires a certain new geometrical calculus, which is completely different from any calculus hitherto known and whose development is reserved for another occasion" (*Phil.*, VII, 59).

[3] See Chap. 7, §5, and Appendix III, §4.

[4] "I am not afraid to say that there is a method of advancing algebra beyond what Viète and Descartes have left us, by as much as they surpassed the ancients" (*Math.*, II, 17; *Brief.*, I, 567).

machines and movements with characters, just as algebra represents numbers or magnitudes. I am sending you an essay that seems to me important."[5]

In this essay, Leibniz develops and refines the idea of "this truly geometrical analysis": "I have found several elements of a new characteristic that is completely different from algebra and that will have great advantages for representing to the mind, exactly and naturally, though without figures, everything that depends on the imagination."[6] This characteristic would allow one to describe with letters of the alphabet not only geometrical figures, but also the most complicated machines and even natural objects like plants and animal structures. Such descriptions, rigorously set out, would be far preferable both to geometrical figures and to verbal definitions; bequeathed to posterity, they would allow the exact reconstruction of a machine. "But its principal usefulness," Leibniz continues, "would consist in the inferences and reasonings that could not be expressed using figures... without multiplying them excessively or confusing them with too many points and lines—inasmuch as we would be obliged to make innumerable unsuccessful attempts—whereas this method would lead us surely and painlessly. I believe that by this means we could handle mechanics just like geometry, and even begin to examine the qualities of materials, since this ordinarily depends on certain shapes of their sensible parts. Finally, I have no hope of our going any further in physics before finding such a shortcut to relieve the imagination."[7]

Leibniz ends this essay by signaling the connection between his geometrical calculus and his logical calculus, which proceeded from the same idea and constituted part of the same general plan: "I believe it is possible to extend the characteristic to those things which are not subject to the imagination; this, however, is too important and too far ranging for me to explain in just a few words."[8] He was evidently alluding to the logical and metaphysical applications of his characteristic.

2. It seems that Huygens was not very impressed by this great plan, for Leibniz raised the point again in his next letter of 10/20 October 1679 and urged Huygens to give his opinion of it. Once again, he boasts of the advantages of his characteristic. Huygens eventually complied in his letter of 22 November: "I have carefully examined what you sent me concerning your new characteristic, but, to speak frankly, I do not see from what

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[5] *Math.,* II, 18-19; *Brief.*, I, 568-9.

[6] Cf. Leibniz to Haak, 6 January 1680/1 (*Phil*., VII, 20), and Leibniz to Bodenhausen (*Math*., VII, 355).

[7] *Math*., II, 21; *Brief.*, I, 571. We recognize here Descartes's principal ideas: the idea of a method that teaches us to control the forces of the mind and to aid the imagination by representing ideas with sensible signs, and the idea of a universal mechanism, which reduces physical qualities to figure and motion and which therefore subordinates physics to mathematics: "Scarcely anything in physical and mathematical investigations can prove to be more useful in aiding the mind and pursuing into their deepest recesses the natures of things, which operate mathematically" (LH XXXV, I, 5, b).

[8] *Math*., II, 25; *Brief.*, I, 575. Cf. p. 322, n. 1, and a letter to Bodenhausen, in which, after having said of his geometrical calculus that "everything that is subject to the imagination depends on this analysis," Leibniz added, "I hope to take a further step toward those things not subject to the imagination, so that all of human reason will be accurately subjected to a certain type of calculus or expressive characteristic. When a conclusion or solution does not follow from what is given, at least the degree of probability afforded by what is given must be determined" (*Math*., VII, 355.). Thus, he explicitly linked his geometrical calculus to the general science, which is, as we know, divided into two parts, the logic of certainty and the logic of probability.

you have shown me how you could base such great hopes on it. Your examples about loci involve only truths that are already well known to us."

Huygens was referring to the definitions of a sphere, circumference, plane, and straight line as locations of points, which Leibniz had set out in his essay to give an idea of his principles. This objection is hardly fair, for Leibniz, in order to establish his geometrical calculus, had to begin with first principles and hence with truths that were already known. These sorts of objections, moreover, are always raised against inventions and new methods, most notably against the infinitesimal calculus.[9]

In any case, Huygens displayed discreetly how little he thought of the geometrical calculus by inviting Leibniz to pursue instead his inquiries concerning the arithmetical quadrature and the roots of equations of degree greater than three. It is clear, however, that he did not understand Leibniz's plan, for he made this objection: "I do not see by what right you are able to apply your characteristic to all these different things that you seem to want to reduce to it, such as quadratures, the discovery of curves through properties of tangents, the irrational roots of equations, Diophantine problems, the shortest and most elegant constructions of geometrical problems, and, what seems to me strangest, the invention and explanation of machines.[10].

Huygens obviously confused the various subjects that Leibniz discussed in his first letter,[11] and Leibniz pointed this out to him in his next reply: "The irrational roots and Diophantine method have nothing in common with this characteristic of situations; thus it is not by that route that I make claim to them." He then applied himself to refuting Huygens's objections: "First, with this calculus, I can express perfectly the entire nature and definition of shape [which algebra can never do][12].... And I can do this for all figures, since they can all be explained by means of spheres, planes, circles, and straight lines, which I have treated in this way.... But machines are nothing but certain figures, so I can describe them with these characters, and I can explain the change of situation that can occur in them, that is, their movement. Second, when we can perfectly express the definition of something, we can also find all its properties."[13]

Huygens, however, was still not converted, for he replied, "Regarding the results of your characteristic, I see that you continue to be persuaded by them, though as you yourself say, the examples should involve more than inferences. That is why I ask for

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9 Huygens only recognized the superiority of the infinitesimal calculus after Leibniz and Bernoulli discovered the properties of the catenary (hanging chain) using it, properties that Huygens had not discovered or even sought (see *Math*., II, 7, 45, 47, 98, 102, 109, 161-2). Cf. Leibniz's letter to Remond, 14 March 1714: "If I have succeeded in bringing eminent men to cultivate the infinitesimal calculus, it is because I have been able to give important examples of its use. Huygens learned something of it in letters from me, disdained it and refused to think there was any mystery in it until he saw some surprising uses of it, which brought him to study it just before his death (*Phil*., III, 611). Leibniz here tacitly compared the fate of his infinitesimal calculus with that of his characteristic, which he discussed later in the same letter (cited p. 395, n. 3).

10 *Math*., II, 27-8; *Brief*., I, 577.

11 See p. 389. These are the same as those he already spoke of in his letter to Galloys of December 1678 (*Math*., I, 183).

12 Here Leibniz gives an example to which we shall return (§5).

13 *Math*., II, 30; *Brief*., I, 580. We recognize here the postulate of universal intelligibility, that is, the principle of reason: all the properties of a thing follow logically from its essence and must be deduced analytically from its definition.

simpler examples, which are apt to overcome my incredulity, for those about loci, I must say, do not seem to be of this sort."[14]

Huygens's demand must have seemed excessive to Leibniz, and not without reason, for what could he have wanted that was simpler than the elementary definitions of a straight line, plane, circle, and sphere? Thus, Leibniz replied rather dryly, like someone who regrets not having been understood and despairs of making himself understood, "To give an example of my characteristic, I chose loci, because I determine everything else through the intersection of them, and because the generation of all other loci depends on the simplest, which I gave. Thus I think I have laid the true foundations."[15]

This is the last mention we find of this plan in the correspondence between Leibniz and Huygens. We can easily guess from this silence that Leibniz gave up the idea of converting his teacher[16] and preferred to discuss other questions on which they understood each other better. He had, moreover, enough other inquiries and studies that could divert him from pursuing this project.

3. He did not, however, abandon the plan, for he talked about it thirteen years later with the Marquis de L'Hospital, in the hope of finding in this disciple a more open and accepting mind, as well as a coworker able to develop the numerous methods he had conceived:[17] "I also have a plan for a completely new geometrical analysis, which is entirely different from algebra and serves *to express situation* in the same way that algebra serves *to express magnitude*; calculations in it are true representations of shape and lead directly to constructions."[18] This sentence and those that follow show sufficiently that this is the same project mentioned in the letters to Huygens. In them, we find the same allusion to the logical calculus: "I shall say nothing to you here about the studies I have made in reasoning mathematically about matters that are entirely removed from mathematics."

L'Hospital did not appear to understand the project any better than Huygens; like the latter, he played the doubting Thomas by demanding more tangible proofs: "What you have told me about your geometrical analysis awakens in me a great curiosity, but I cannot arrive at an accurate idea of it without having first seen some examples."[19]

Leibniz himself implicitly recognized the legitimacy of these reservations, for he wrote a little later: "I do not dare yet to publish my plans for a characteristic of situations, for unless I make it plausible with examples of some consequence, it would be taken for a

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[14] Huygens to Leibniz, 11 January 1680 (*Math*., II, 35; *Brief.*, I, 584).

[15] Leibniz to Huygens, 26 January 1680 (*Math*., II, 36; *Brief.*, I, 585).

[16] As Gerhardt (*Math*., II, 5) along with many others has noted, Leibniz, with his diplomatic tact, never insisted on ideas or projects that his correspondents did not understand or favor. With Xenocrates he used to say, "he doesn't have a handle on this subject." Cf. König, *Appeal to the Public*, p. 87 (see Note XVI).

[17] Cf. Leibniz to L'Hospital, 28 April 1693: "If I were as capable of completing these methods as I am of planning them, we would no doubt make tremendous progress" (*Math*., II, 236).

[18] *Math*., II, 228 (1693). Cf. Leibniz to Arnauld, 23 March 1690, in which Leibniz, after summarizing his philosophy, adds: "And I will not speak further of a completely new analysis proper to geometry and entirely different from algebra" (*Phil*., II, 137); Leibniz to Bernoulli, 24 September 1690 (we note that from his very first letter Leibniz is eager to inform Bernoulli about his plan): "I have in mind a properly geometrical analysis, completely different from algebra, which does not proceed through equations and which will have noteworthy applications. For the symbolisms in use to this point

[19] *Math*., II, 234.

mere fantasy. Nevertheless, I see in advance that it could not fail."[20] Since, however, on the one hand Leibniz did not have the leisure to work out the essays or examples that would be needed to prove the value of his plan, and on the other L'Hospital was more interested in other, more technical, subjects that Leibniz discussed with him (notably the infinitesimal calculus, about which L'Hospital was soon to publish the first treatise), the geometrical calculus was once again set aside, and there was no further consideration of it.[21]

4. Fortunately, we possess several "samples" of this calculus,[22] which allow us to gain a more accurate and complete idea of it than L'Hospital or even Huygens could have formed on the basis of the letters cited above. These sketches seem to belong to two main periods, which are quite distinct. The more important, entitled *Geometrical Characteristic*, is dated 10 August 1679,[23] which shows that Leibniz submitted his project to Huygens quite soon after developing it; indeed, the essay he sent with his letter of 8 September is nothing but a summary of this study.

[20] Leibniz to L'Hospital, 27 December 1694 (*Math*., II, 258). Cf. Leibniz to Remond, 14 March 1714 (cited in the following note).

[21] It is, however, mentioned later in two passages that we want to cite, for they give some interesting clues about the biography and psychology of Leibniz: "The Abbé Le Torel tells me that you have spoken about my geometrical calculus. This is presumably what I call my calculus of situations. I am myself irritated that I have not been able to develop to my own liking a thought that seems to me of some consequence. But nothing is more tedious than work done in isolation, which can be discussed with no one. Spoken communication between those involved in the same research is one of the best seasonings to intrinsically dry meditations. However, I see no likelihood of this, unless I someday find a young man who is suited for taking up my views." Leibniz to L'Hospital, 13/23 March 1699 (*Math*., II, 334). Two years later, after having spoken of his binary arithmetic (see Appendix III), he writes: "My *analysis of situation* seems even more curious.... I must someday set myself to laying out its elements. A very clever gentleman from among my friends, and a remarkable geometer besides, began this task, but his death has deprived us of what he might have done. I should require the assistance of someone like him, who was deep, had a longing for the truth, and was of a very mild and rational humor. The combination of these qualities, however, is quite rare." Leibniz to L'Hospital, 26 September 1701 (*Math*., II, 342). Toward the end of his life, ill and exhausted, Leibniz wrote these sad lines: " I discussed my general symbolism [*specieuse générale*] with the Marquis L'Hospital and others, but they paid no more attention to it than if I had related a dream to them. It would have been necessary to have supported it with some tangible application, but for this I would have had to construct at least a part of my characteristic, and this is not easy, especially in my present condition and without the conversation of persons who could stimulate me and assist in work of this nature." Leibniz to Remond, 14 March 1714 (*Phil*., III, 612). Cf. his first letter to Remond of 10 January 1714 (cited at the end of Chap. 4). We observe that Leibniz continued to link his geometrical calculus to his universal characteristic, and that he never succeeded in setting it out definitively, for lack of intelligent and cooperative assistants, whom he always sought and so rarely found. (See Chap. 5, §24, and Note XV.)

[22] "I have some samples of it, which will keep this view from being lost in case I am prevented from seeing it through." Leibniz to L'Hospital, (*Math*., II, 229). He had already written in his essay of 1679: "But as I am not aware of anyone else ever having had the same idea, I fear that it may be lost if I do not have time to complete it, and I am adding here an essay, which seems to me of some importance and will suffice at least to make my plan more credible and easier to understand. The point of this is that, should some accident prevent its realization, this essay will serve as a monument to posterity and allow someone else to finish it" (*Math*., II, 22).

[23] *Math*., V, 141-171. The same title is found on some unpublished drafts, one of which is dated January 1677 (*Bodemann*, 286). This is the same period in which Leibniz was trying to construct his universal characteristic, his rational language, and his logical calculus. Cf. *An Example of the Philosophical Language Displayed in Geometry*, January 1680 (LH IV 6, 10b).

It is appropriate to this link this piece with *On the Analysis of Situation*,[24] which is related to it in content. We find there again the idea of applications to mechanics; the geometrical calculus is presented as a new invention, and it rests on the definition of similarity, which Leibniz communicated to Galloys in 1677 as a recent discovery.[25] *On Euclid's Elements*,[26] a critical analysis of the definitions, axioms, and postulates of Book I of Euclid, appears to be a preliminary study for the *Geometrical Characteristic*; we shall see (as we already know) that the analysis of axioms and definitions is an indispensable step in the development of a characteristic. Moreover, Leibniz himself stated that in order to establish his calculus of situation, he would analyze the demonstrations of Euclid,[27] and among his papers we indeed find a fragment entitled *Demonstration of Euclid's Axioms*, dated 22 February 1679.[28] *On Euclid's Elements* is thus probably from the same period.

We cannot say the same for a study dedicated to the properly geometric analysis and the calculus of situation.[29] This must have been composed in 1697 or 1698, for the draft was sent to Bodenhausen in January 1698.[30] It seems that in this period Leibniz took up his plans for a geometrical characteristic once more at the request of Bodenhausen.[31]

Finally, it is important to relate to the preceding works, brought together by Gerhardt in *Math*., V, a number of others that he left to *Math*., VII: *On Construction*; *Example of an Enlightening Geometry*;[32] *Preface to the Key of Mathematical Secrets*; *Mathematical Inventory* (which seems intended for the encyclopedia);[33] *Universal Mathematics*; *New Advancement of Algebra*; *On the Origin, Progress and Nature of Algebra*; *Mathematical Foundations*; and especially, *Metaphysical Foundations of Mathematics*, which coming after 1714 contains Leibniz's definitive thoughts on the philosophy of mathematics.[34] To these we can join two unpublished fragments, *General Mathematics*[35] and *Idea for a*

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[24] *Math*., V, 178-183. It was undoubtedly from Leibniz that Riemann borrowed the title of his own *Analysis Situs* (*Gesammelte Werke*, 448, Leipzig, Teubner, 1876), by which he meant, however, a science somewhat different from what Leibniz understood by the term.

[25] See n. 81.

[26] *Math*., V, 183-211.

[27] "As far as is necessary and reasonable, we shall reduce the demonstrations of Euclid (as they are set out by Clavius) to the calculus of situation, so that we may better establish the elements of such a calculus (LH XXXV, I, 3; *Bodemann*, 285). Cf. LH XXXV, I, 3, e; I, 12; I, 14, d.

[28] LH XXXV, I, 2. Cf. *First Principles of Geometry* (LH XXXV, I, 5) and *Primary Propositions of the Elements* (LH XXXV, IV, 13, d).

[29] *Math*., V, 171-178. The signs and notations differ, moreover, from those employed in the other studies.

[30] "This essay was sent to the honored Baron Bodenhausen in Florence in January 1698" (*Bodemann*, 286; cf. *Math*., V, 140).

[31] Leibniz wrote to him: "Right now it is a little difficult for me to send you an example of my new analysis of situation, for I must completely rethink it from the beginning; nonetheless, I shall soon dedicate myself to the task"; and on 6 Dec. 1697, "concerning my calculus of situation, I am not able to display it in a finished form so long as it remains a mere idea and does not receive any application" (*Math*., VII, 362, 393).

[32] This text would be prior to 1687, if it is the one alluded to in letters to Foucher and Arnauld (cited p. 304, n. 1).

[33] Cf. the unpublished fragment LH XXXV, I, 26, which seems to be another preface to *Mathematical Inventory*.

[34] In its opening allusion is made to an article that appeared in the *Acta Eruditorum* in 1714 (*Math*., VII, 17).

[35] LH XXXV, I, 9 a.

Book Whose Title Will Be: New Elements of Universal Mathematics.[36] From this set of essays and sketches, we can, in spite of inconsistencies due to differences in time, disentangle the main ideas that inspired the project of a geometrical characteristic and guided its development.

5. In the first place, Leibniz was convinced of the insufficiency and imperfection of algebra as the logical instrument of geometry. In December 1678 he had already written, "I am searching for practically nothing more in geometry than a way of finding right away elegant constructions. I see more and more that algebra is not the natural way of arriving at these, and that it is possible to establish another characteristic proper to lines and natural for linear solutions, whereas algebra is common to all magnitudes."[37] Algebra, in fact, is only one branch of the characteristic: it is the characteristic of magnitudes or indeterminate numbers;[38] it does not express "situation, angles, or movement directly."[39] Algebra is forced to translate relations of situation into relations of magnitude; it follows that analytic geometry expresses geometrical facts only in a complicated and roundabout way. To represent *one* point (that is, one element of situation), *two* or *three* magnitudes are required (the coordinates of the point).[40] Figures are given only indirect and artificial definitions; for example, when we say that "$x^2+y^2=a^2$ is the equation of a circle, we need to explain what x and y are."[41] This shows that the analytic definition of a figure is necessarily relative to the coordinate system adopted and, what is more, to the choice of axes or frames of reference. Consequently, the equation of one and the same shape can vary infinitely; it contains arbitrary constants that depend on the choice of axes and the origin and which unnecessarily complicate the equation. To arrive at a simple and clear form of the equation, we are obliged to choose particular axes and to assign the figure some unusual position. The formulas of analytic geometry are not intrinsic but extrinsic; in other words, they define a figure not by its internal relations but by its relations to some reference system, which is arbitrarily selected. It follows that algebra does not translate the geometrical construction of given figures, nor does it provide "the most beautiful" (that is, the simplest and most natural) constructions of

[36] LH IV 7B, 6 Bl. 9-12.

[37] Leibniz to Galloys (*Math*., I, 183). Cf. Leibniz to Tschirnhaus, end of 1679 (after their meeting in Hanover): "Algebra is not the true geometrical characteristic, but mine, which I showed you, approaches it more nearly" (*Brief*., I, 413); another letter to Tschirnhaus (also from 1679), after having said that the best signs are images: "Since algebra does not offer this when applied to geometry, I therefore prefer my geometrical calculus, which I showed you (*Math*., IV, 481; *Brief*., I, 405); Leibniz to Haak, 6 January 1680/1: "I will add even further that algebra itself is not a true geometrical characteristic, and that a very different one must be invented, which I am certain will be more useful than algebra in applying geometry to the mechanical disciplines" (*Phil*., VII, 20). "If anything is clear to me, it is that a true geometrical analysis has yet to be proposed, and that the calculus we now possess is more numerical than geometrical, for it is not points that are usually denoted by the letters of this calculus (as would have to be the case in a geometrical calculus), but magnitudes, i.e., indefinite numbers. Thus, magnitude is directly represented in this calculus, and position or shape only indirectly and circuitously" (LH XXXV, I, 5, b).

[38] See Chap. 7, §2. *On the Analysis of Situation*: "What is commonly known as *mathematical analysis* is an analysis of *magnitude*, not *situation*; and so it indeed pertains to arithmetic directly and immediately, but it is applied to geometry only circuitously" (*Math*. , V, 178).

[39] Appendix to the letter to Huygens of 8 September 1679 (*Math*., II, 20; *Brief., I, 570).*

[40] LH XXXV, III, B, 18a.

[41] Leibniz to Huygens (*Math*., II, 30; *Brief*., I, 580).

unknown figures.[42] It always brings in auxiliary magnitudes that are foreign to the figure and have no other role than to connect it with the coordinate system; it is, as it were, a cumbersome scaffolding, which conceals and unnecessarily complicates figures.

Finally, the reduction of relations of situation to relations of magnitude presupposes the fundamental theorems of elementary geometry, notably those of Thales and Pythagoras.[43] As a result, analytic geometry depends on synthetic geometry; it neither pushes to completion the analysis of geometrical concepts nor rests on axioms that are truly primitive; it continues to rely on intuition or the imagination; in a word, it is not autonomous and does not possess the logical perfection that befits a purely rational science.[44]

6. At the same time, Leibniz was not unaware of the weaknesses and shortcomings of the synthetic method in geometry. Intuitive definitions or descriptions lack precision and inferences based on intuition lack rigor. The proof of this is that synthetic geometry itself is obliged, in order to be able to reason about figures, to take into account the abstract relations of magnitude that determine their form. For example, to express the fact that three points A, B and C lie on a straight line, we would write the equation AB+BC=AC.[45] Furthermore, strictly speaking, we do not reason about the imperfect and inexact figures that the chalk traces on the blackboard, but about ideal figures of which these are only rough images. But what accounts for the absolute precision of these images, if not the abstract relations that a geometer either discovers in them or introduces with his thought? In reality, even when we seem to be making an appeal to intuition, it is not the sensible figure that is invoked but the intelligible relations that are embodied in it, or which are understood to be there on the basis of hypotheses. We know how dangerous it is to rely

[42] "It follows from this that quite complicated algebraic calculations often arise from simple geometrical descriptions, and conversely, that it is difficult to derive easy constructions from algebraic calculations" (LH XXXV, I, 5, b).

[43] *Geometrical Characteristic*, §5 (*Math*., V, 143); *On the Analysis of Situation* (*Math*., V, 179). Cf. *New Essays*, IV.vii.19; *Primary Propositions of the Elements* (LH XXXV, VI, 13d). The theorem of Thales on similar triangles relates similarity (identity of form) to the proportionality of sides (a relation of magnitude). The Pythagorean theorem expresses the hypotenuse of a right triangle as a function of the two other sides; thus it furnishes, on the one hand, a linear measure of a right angle (i.e. a reduction of angles to lengths), and on the other, a formula for a linear element (i.e. the expression of one length as a function of its projections onto the coordinate axes). Cf. Cournot, *De l'origine et des limites de la correspondance entre l'Algèbre et la Géométrie*, Chap. VIII, §78 (Paris, Hachette, 1847).

[44] Leibniz to Bodenhausen (after 1690): "I have contemplated formalizing my calculus of situation, since up till now we have had only a calculus of magnitude, with the result that our analysis has not been complete but has depended on the elements of geometry" (*Math*., VII, 355). "It is also clear that the algebraic calculus does not express everything that must be taken into consideration, but presupposes much from elementary propositions and the inspection of figures. It follows that the analysis comes, as it were, to an abrupt stop halfway home and does not reach its goal; and so it is not capable of all the transformations that the nature of the thing furnishes" (LH XXXV, I, 5b).

[45] *Geometrical Characteristic*, §4 (*Math*., V, 142). Curiously enough, it was the generalization of this formula that gave rise to the modern geometrical calculus. Mobius began by extending it to three points situated in any order on a straight line; Grassmann then attempted to extend it to any three points on a plane or in space, stripping it, of course, of its quantitative or *metric* sense (Preface to his *Ausdehnungslehre* of 1844). See Appendix V, §1.

on intuition in geometrical proofs; in no case do these proofs owe their validity or probative force to intuition.[46]

Leibniz thus recognized that analytic geometry has at least one advantage over synthetic geometry: To solve problems, the latter relies almost exclusively on the imagination, which is not, strictly speaking, a method, or at least not a general and reliable method. Synthetic geometry requires guesswork and a kind of wholly empirical and individual insight; moreover, we find a solution (when we do find it) only after a series of attempts made more or less at random, which tire the imagination and obscure the figure.[47] Finally, the solutions we do find are most often particular, depending on some accidental detail in the construction or on some happy artifice that only works in special cases, so that for other cases we need to put the imagination to work once again, repeat the same efforts and set off in quest of some further construction. Analysis, on the contrary, proceeds via long and twisting, but determinate paths;[48] it leads surely, in a mechanical and, as it were, inevitable way to the solution, for it is a sort of Ariadne's thread, which allows us to find our way through the labyrinth of intuition.[49] Nevertheless, the general procedures of analysis are not always the simplest and most natural; they are often artificial and needlessly complicated. This is why Leibniz dreamt of a method that would bring together the advantages of analysis and synthesis, without incurring the inconveniences of either.[50]

7. As we have seen, however, there is at the heart of the synthetic method itself a natural and spontaneous, indeed almost unconscious, sort of analysis, which rests on the abstract relations of figures and on their necessary logical connections. Leibniz conjectured that ancient geometers possessed an analysis of this type, which served as a method of discovery and of demonstration, and which took the place for them of analytic geometry, allowing them to solve problems for which we employ algebra.[51] To establish the

[46] See *New Essays*, IV.i.9: "It is not the figures that serve as proofs for geometers.... The force of the demonstration is independent of the figure drawn" (against Locke's empiricism). Cf. *New Essays*, IV.xii.6 (cited in §17 of this chapter).

[47] *Math*., II, 21 (cited in §1), and LH XXXV, I, 5b (cited in n. 52).

[48] "By an analysis, that is to say, by determinate means." Leibniz to Huygens (*Math*., II, 21). See p. 406, n. 1.

[49] Cf. Chap. IV, §5

[50] Leibniz to Bodenhausen: "It is certain that by reducing everything from position to magnitude, algebra thereby often makes things quite complicated. It does have the advantage that it always (in ordinary geometry) can come to some conclusion, although occasionally it takes a very circuitous route. It is as if someone wanted to solve all problems of the same degree using the same given circle and the same constant parabola, which though always possible, is not always best" (*Math*., VII, 362).

[51] He cites as examples of it the *givens* of Euclid, and the linear, plane and solid loci of Apollonius and Pappus (*On the Analysis of Situation*; *Math*., V, 179). Cf. *On Construction*: "From this I can see that geometry, although it is a science which has been subordinated to the algebraic calculus, nonetheless has a type of analysis peculiar to itself, by which properly geometrical theorems may be demonstrated , and by which the most involved constructions may be carried out in the end using straight lines, with the calculation shortened as much as possible. The ancients seem to have recognized and used this properly geometrical analysis, for in their writings I seem to find traces of something other than algebra, where there is no concern with numbers (*Math*., VII, 254). Leibniz to Boden hausen: "I am myself of the conviction that in problems of common geometry, the method of the ancients, and the analysis of which certain traces are found in Pappus, have definite advantages over algebraic analysis; thus I also believe, in contrast to my most distinguished correspondent, that there still remains a properly geometrical analysis, completely

characteristic for geometry, it would suffice to formulate and systematize this natural analysis and to translate it using appropriate symbols.

Above all, we must must complete the analysis of the elements of this science, in such a way that we obtain its simple concepts and first principles. Once this is done, we would express the concepts by signs and the principles by formulas, so that we would have reduced the entire science to symbols, beginning with its elements. Thereafter, instead of being applied after the fact and from without to a geometry that was complete and already established, this analysis would, so to speak, arise and develop with it; it would remain intimately tied to it and would translate all propositions in a direct and adequate manner. It would allow us, therefore, to demonstrate all geometrical truths, including the axioms, whence it would begin.[52] In geometry, as in every other science, the analysis of concepts and the demonstration of axioms are united in a single investigation and together serve to reveal the foundations of the cha racteristic. This analysis would render the elements of geometry independent of both the science of magnitude and intuition; consequently, it would free geometrical demonstrations from calculations on the one hand, and from figures on the other. It would transform these demonstrations into purely logical inferences, which could be expressed by means of simple formal deductions. This is why, when Leibniz proposed in January 1680 to give an example of his philosophical language, he first thought of applying it to geometry.[53]

8. It is chiefly in the solution of problems that the proposed characteristic would surpass analytic geometry. In fact, analytic geometry demands a double translation: first from the

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different from algebran, which is in many respects far more concise and useful than algebra (*Math*., VII, 358-9). Leibniz to Foucher, 1687: "I will tell you again that the ancients had a certain geometrical analysis completely different from algebra.... It has uses quite other than those of algebra, and although it gives way to it in certain contexts, it surpasses it in others (*Phil*., I, 395). Cf. *New Essays*, IV.xvii.13; LH XXXV, IV, 13g; *Phil*., VII, 298 (cited in n. 18).

[52] "Algebra is obliged to presuppose the elements of geometry, whereas this characteristic pushes the analysis to its end (*Math*., II, 21). "By means of a type of calculus, we thus find everything that geometry has to offer, all the way down to its elements, in an analytic and determinate manner. Algebra, which presupposes these elements, does not carry the analysis to its end, as this new characteristic does" (*Math*., II, 26). "Having trimmed many things away, I see that I have at last come to those things which are most simple, for I am presupposing nothing from elsewhere but am able to prove everything from the appropriate characters." *Geometrical Characteristic*, §8 (*Math*., V, 144). Cf. Leibniz to Bodenhausen: "As I see it, however, the elements themselves must issue from the calculus" (*Math*., VII, 355). "But if this analysis is to be applied to the direct expression of situation and extended all the way back to first principles, whence the elements of geometry themselves will be demonstrated, then all those things which are now discovered only with complex constructions of figures and by wearying of imagination, could be described and established directly by a certain kind of combinatorial calculus" (LH XXXV, I, 5b).

[53] *An Example of the Philosophical Language Displayed in Geometry*: "Since I will have first explained the elements in this way, the steps to all other things will not be difficult. I shall, however, add nothing to this calculus concerning magnitudes, sums, differences, composite ratios of ratios, potentials or sums [i.e. intergrals], or any other things that arithmetic and geometry have in common. Rather, I shall restrict myself to points, lines, angles, intersections, contacts and motions, and I shall show how calculating [*calculares*] or mixed expressions are reducible to lines. The rewards of this will be very great, since in this way we will be able to carry out the most subtle geometrical reasoning, without the aid of paper, effort, or calculation, using our imagination and memory alone" (LH IV 6, 10b). In this fragment, Leibniz defines what we call *projective* geometry, as opposed to *metric* geometry. Earlier he writes, "I shall reduce everything to straight lines." This is precisely what Staudt accomplished in his *Geometrie der Lage* (1847), where he expresses all inferences verbally, using no calculations or figures.

geometrical givens into equations, and then from the algebraic solution back into geometrical terms, and this double translation is often difficult and painstaking.[54] Moreover, the algebraic inference by which we derive unknowns from givens is in general completely different from the geometrical arguments and constructions by which we can obtain the solution directly, so the calculation does not preserve at all the natural and logical flow of thought. Leibniz illustrated this in an appendix to his *Geometrical Characteristic* by solving the same problem using, in turn, algebra and pure geometry: Construct a triangle, given the base, height, and angle of the apex.[55] This example shows strikingly that the construction provided by algebra is completely different from the construction obtained by synthesis, that it is much more complicated, and finally, that it is artificial and circuitous, in no way corresponding to the natural connection of ideas or the intuitive properties of the figure. What Leibniz sought, by contrast, was an analysis that does not lose sight of the figures, that follows step for step geometrical reasoning, and that always gives a faithful translation of the constructions suggested by the intuitive, synthetic method: "In this new calculus, the mere statement of the problem would be its calculation, and the final calculation would be the expression of the construction."[56]

To what do we attribute the mismatch between algebra and geometry, and hence the imperfection and needless complication of analytic geoemtry, and the divorce of calculation and construction? It is to the fact that algebra, insofar as it is the science (or rather the logic) of magnitudes, is unable to express situation except by reducing it forceably to magnitude.[57] Consequently, it is clear that in order to set up a properly geometric characteristic, we must invent a new analysis of situation (*analysis situs*) that directly expresses relations of position, and hence configurations and constructions. Leibniz sees no theoretical obstacle to this, for if algebra is nothing more than an application of the universal characteristic to numbers and magnitudes, nothing prevents us from applying the same characteristic to geometry, or employing letters and signs analogous to those of algebra to represent points, relations of place, and even qualities.[58]

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[54] "The translation of geometrical problems into algebra, by reducing position to magnitude, is often quite difficult, both in the way we must set up the problem as a calculation and even more so in the way we derive a construction from the finished calculation." Leibniz to L'Hospital, 1693 (*Math*., II, 228). Cf. Leibniz to Jacob Bernoulli, 24 September 1690 (cited p. 394, n. 2) and LH XXXV, I, 5b (cited p. 400, n. 1).

[55] *Math*., V, 168-171. In this fragment Leibniz makes use of the ambiguous signs of the *Method of Universality* (LH IV 5 Bl. 10). See Chap. 7, §10.

[56] Leibniz to L'Hospital, 1693 (*Math*., II, 229). Cf. Leibniz to Huygens: "By focusing on visible figures, this new characteristic cannot but help provide at once the solution, the construction, and the geometrical demonstration, and all of this by a natural process of analysis" (*Math*., II, 20-21). And elsewhere: "This characteristic will contribute greatly to finding elegant constructions, since the calculus and the construction are found in it at precisely the same time (*Ibid*., 30-31). Leibniz to Tschirnhaus, May 1678: "Formulas can in fact be devised* that express situation and the drawing of lines and angles without having recourse to magnitudes, and with the help of which we shall find constructions more easily and elegantly than with the calculus of magnitudes" (*Math*., IV, 460; *Brief*., I, 379-80). "For this calculus treats of magnitude , while geometry treats of both magnitude and situation; the consideration of situation, however, has its own shortcuts, which cannot be expressed through a consideration of magnitude alone unless considerable energy is expended. Thus, for the sake of constructions, I think that some sort of analysis remains to be found that is purely geometrical and quite different from algebra" (LH XXXV, IV, 13f).

[57] "Dragged by the neck." Leibniz to Jacob Bernoulli (cited p. 394, n. 2).

[58] *On the Origin, Progresst, and Nature of Algebra*: "There are certain calculi, quite different from those hitherto in vogue, in which the signs and characters do not stand for quantities or numbers, either

9. Leibniz tried several times to give a philosophical analysis of the notion of situation: *position* is that which distinguishes objects that present no intrinsic distinction, and *situation* is position in space (i.e., in the order of coexistence), just as instants are positions in time.[59] But this definition is unsatisfactory, for it risks confusing position with magnitude.[60]. Elsewhere, Leibniz remarks that situation is a relation, such that all things that have a situation with respect to a given thing have, by the same token, a situation among themselves.[61] Pursuing this idea, he finds first that situation implies an order , but an order that is entirely relative and even reversible, as in the order among the points of a line, which can be conceived as starting at either end point.[62] He then finds that situation (or extension) involves the simultaneous perception of a plurality of objects. However, this is not enough, for we need to perceive a certain relation among these objects; and this relation must be uniform (i.e., homogeneous); it must be identical, or at least similar, among all the objects perceived together, whatever their qualitative and sensible differences. On the other hand, situation also implies a certain distinction, even between the most similar objects, such as the parts of a homogeneous body. Finally, situation is independent of place or absolute position, for the same objects can have the same relative situation whether they are here or there.[63] All these characteristics of the relation of situation render a logical analysis of this notion extremely difficult, and Leibniz does not seem to have succeeded in finding a definition of it.

In lieu of situation, he defined a *point*, which is an elementary and simple situation. It is that locus of which no other locus can be part; in other words, it is a locus X in which no other locus Y can be contained without coinciding with it or being identical with it.[64] This definition, like all definitions, expresses a reciprocal property; that is to say, if a locus X is such that any locus Y contained in it necessarily coincides with it, then X can only be a point.[65] We note the abstract and purely logical character of this definition; it is analogous to those Leibniz gave of the notion of an *individual* or of the number *one*.[66] The definition of space is in a sense the counterpart of this definition: Space is the total or complete locus, or the locus containing all other loci. In particular, it is the locus of all

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definite or indefinite, but for entirely different things, such as points, qualities, and relations. For example (to say nothing about the calculus of the figures and moods of logic, where the letters stand for the quantities and qualities of propositions), there is a certain peculiar analysis and a purely geometrical calculus that I have devised, which is completely unlike that hitherto received; it directly expresses not quantities but situation, for the algebraic calculus distorts situation into magnitude, and thus leads it off into obscurity" (*Math*., VII, 207). Cf. *Geometrical Characteristic*, §7; *Math*., V, 144).

[59] LH XXXV, I, 3c. This definition rests implicitly on the principle of indiscernibles.

[60] See Chap. 7, §11.

[61] *First Principles of Geometry* (LH XXXV, I, 5a).

[62] LH XXXV, I, 9e.

[63] LH XXXV, I, 5c.

[64] "A *point* is a simple locus, or a locus in which there is no other locus. Thus, if B is in A, then A∞B" (LH XXXV, I, 3a). Cf. LH IV 7C Bl. 179; and *Metaphysical Foundations of Mathematics*: "A *point* is the simplest locus, or the locus of no other locus" (*Math*., VII, 21).

[65] "If from the supposition that B is in A, it is understood by this fact alone that A and B coincide, then A is called a *point*. Consequently, if B is in A and it follows that A∞B, then A is a point. And if B is in A and A is a point, then A∞B" (LH XXXV, I, 5d). Cf. *A Purely Geometrical Analysis*, §9, 1698 (*Math*., V, 173-4).

[66] See Chap. 8, §11. This is precisely the definition given by Peirce of an *individual* or (logical) *point*. See Schröder, *Algebra der Logik*, vol. II, §47 (Leipzig, Teubner, 1891).

points, so that if we designate the locus of the point X by $\underline{X}$, space will be the locus of any arbitrary point P, i.e. $\underline{P}$ (or the *set* of all points).[67]

Leibniz next tried to define the situation of a point, that is, the mode of determination of its distance with respect to other points, whose mutual distances (and hence relative situation) are fixed.[68] In particular, in the plane, the situation of a point is determined by its distance with respect to three fixed points (not in the same plane). But these considerations rest on ideas that are far too complex, namely ideas of magnitude (distance), straight line, and plane.

10. As with space, all figures are conceived as sets of points. To determine the position of a (solid) figure in space (to fix its position, as we say), it is necessary and sufficient to give three of its points (not in a straight line).[69] The form of a figure, on the other hand, is determined by the relations among its different points, and is given once we assume certain notable points, which serve as the basis for its construction. In particular, every (algebraic) line is determined once we know a certain (finite) number of its points (equal to its degree). We can therefore consider all figures and constructions as *combinations* of points, or as collections of lines. The simplest line is the *straight line*, determined by just two of its po ints; moreover, in projective geometry, which Leibniz anticipated, all constructions can be reduced to the drawing of straight lines. These *linear* constructions must then be expressed by means of a *linear* analysis, which directly represents relations of situation, independently of any *metric* notion (i.e., of number or magnitude) and of any algebraic calculus.[70] This would be a *descriptive* geometry (in the true sense of the word), expressing the construction of figures, and hence all of their properties, in terms of *intrinsic* relations among their points, without appeal to any coordinate system. Thus, in the geometrical calculus, an arbitrary letter or symbol would no longer represent magnitudes or numbers (as in algebra), but points and combinations of points.[71]

With figures defined in this way, all the inferences carried out on them by synthetic geometry would be translated into formulas, thus giving rise to a *calculus of situations*, which Leibniz opposes to the *calculus of magnitudes*. He enumerated the principal operations and relations of this calculus as follows: We would construct figures by means of sections and motions;[72] then we would study in them, besides magnitude (emphasized too much by classical geometry) and equality (of magnitude), relations of similarity,

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[67] "*Absolute space* is the most complete locus, or the locus of all loci" (*Math*., VII, 21). "*Space* is the locus of all points; if P is any arbitrary point, space will be $\underline{P}$ (LH XXXV, I, 5a). This is precisely the view of Peano, for whom space is the class of points, such that the term *point* (in a generic sense) is synonymous with *space* (in a collective sense).

[68] "The *situation of a point* is the mode of determining the distance of this point from any others whose distance among themselves is determined" (LH XXXV, I, 8; cf. IV, 13e).

[69] LH XXXV, IV, 13e. Leibniz believed that four points were necessary to fix a surface and five to fix a solid. This is wrong; no more are needed than to fix a simple line. He did, however, write on this draft: "These things need to be examined more carefully."

[70] *Geometrical Characteristic*, §6 (*Math*., V, 143). Cf. LH XXXV, IV, 13g.

[71] "But in a truly geometrical calculus based on points, the very formula that is defined or discovered with this calculus ought to be an expression of the description or construction itself" (LH XXXV, I, 5b). Cf. the passage from the same fragment cited in n. 37; and LH XXXV, III, B, 18a.

[72] If we substitute for the idea of motion, the more general one of *projection*, we then have the two fundamental operations of projective geometry.

congruence, coincidence, and determination.[73] In short, the geometrical characteristic would have its foundation in the application to geometrical figures of the categories of the universal mathematics, each of which provides the object for a special calculus, or a different algorithm.[74]

11. We have seen that in his universal mathematics, Leibniz consistently distinguished quantity and quality, or magnitude and form,[75] and that in geometry he opposed magnitude to situation, a distinction which coincides with the preceding one. Thus, in order to analyze geometrical figures completely, both these elements must be taken into account; and this is precisely where algebra falls short, since it considers only magnitude (measured with numbers) and cannot express form, the purely geometric element, except by translating it into relations of magnitudes. Two figures having the same magnitude are equal, whereas two figures having the same form are *similar*.[76] The geometrical calculus must therefore consider not only the equality of figures, but also and, above all, their similarity; the theory of similarity is thus the foundation of the true analysis of situation.[77]

This theory of similarity, conceived as a primitive relation or *category* of geometry, had to be created from scratch. The reason geometers had neglected the relation of similarity, or had subordinated it to the relation of equality, was that they lacked a clear and precise definition of this notion. We could say simply that similarity consists in the identity of form, but this would be to define the obscure by the more obscure, as the scholastics were wont to do.[78] On the contrary, we gain a clear idea of *form* only when we have defined *similar* figures.[79] According to Leibniz, those things are similar which are indistinguishable when each is considered separately.[80] Similar objects can differ only in magnitude; hence, magnitude is what distinguishes similar things, and it can only be discerned by "comperceiving" these objects, or by comparing them in intuition.[81] This

[73] *On the Calculus of Situations*: "In the calculus of magnitudes, we form magnitudes when we add, multiply, compound ["multiply" designates the multiplication of a magnitude by a number; "compound," the multiplication of one magnitude by another], and work out reciprocals; when we compare them using ratios and other relations, progressions, inequalities, and equations. With situation, in the same way, we form extended things through sections and motions; then we compare them, considering in them, besides magnitude, similarity, congruence (where equality and similarity come together), coincidence and determination" (LH XXXV, I, 15). Cf. Chap. 7, §§9 and 10.

[74] See Chap. 7, §§14 and 16.

[75] See Chap. 7, §4.

[76] See Chap. 7, §11.

[77] *On the Analysis of Situation* (*Math*., V, 179). In *Foundations and Examples of the General Science*, the plan for geometry is already mentioned in these terms: "Geometry, in which magnitude and similarity of situation are united" (*Phil*., VII, 59). We shall see how Leibniz, indeed, applies to geometry these two abstract mathematical categories: "We shall reduce equality to congruence, and ratio to similarity" (LH IV 6 Bl. 10b).

[78] *On the Analysis of Situation* (*Math*., V, 180).

[79] This procedure is in complete conformity with the tendency of modern mathematicians, who define mathematical entities in terms of their identity conditions. Burali-Forti calls this a "definition by abstraction." See his article in *Bibliothèque du Congrès international de Philosophie*, vol. III.)

[80] "Thus, if two things are similar, they cannot be distinguished when each is considered by itself." *Example of an Enlightening Geometry* (*Math*., VII, 276).

[81] Leibniz to Galloys, 1677: "For example, there is no one who has satisfactorily defined what it is to be *similar*.... After having tried this to so, I found that two things are perfectly similar when we can

comparison can be either immediate or mediate: In the first case, the two objects to be compared must be present at the same time; in the second, they are each compared to the same third object, which serves as a standard of measurement.[82] In the end, however, this mediate comparison reduces to an immediate comparison or comperception.

It is for want of this general, philosophical notion of similarity that geometers have defined it in terms of the equality of angles, or in terms of the proportionality of homologous lines. These are only particular derived properties, and in any case *metric* properties, which interpret form in terms of magnitude.[83] An even less appropriate definition is that given of similar triangles, which defines them as having equal angles and proportional sides, for this is redundant and implies a theorem. From his general definition of similarity, by contrast, Leibniz immediately deduced: 1) that two triangles having equal angles are similar; 2) conversely, that two similar triangles have equal angles; 3) that two similar triangles have proportional sides; and 4) conversely, that two triangles with proportional sides are similar. Finally, from this, he deduced the theorem implied by the classical definition: triangles having equal angles also have proportional sides, and vice versa.[84]

The same definition allowed Leibniz to establish immediately, and almost intuitively, the fundamental properties of similar figures: circles are proportional to the squares of their diameters, and spheres to the cubes of their diameters. More generally, in similar figures, lines, surfaces, and volumes are proportional, respectively, to the first, second, and third powers of their homologous sides (or dimensions).[85]

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discern them only by compresence... [an example follows]... not by memory but by compresence..., for the magnitudes cannot be retained. If everything in the visible world were diminished by the same proportion, it is clear that no one would be able to discern the change" (*Math.* , I, 180). In these last lines, Leibniz poses and answers the problem of the indiscernibility of sensible worlds, much discussed lately, and resolved differently, by Renouvier, Delbœuf, and Lechalas. See *Étude sur l'Espace et le Temps* by the last mentioned.

[82] *Example of an Enlightening Geometry* (*Math*., VII, 276). Cf. *Geometrical Characteristic* (1679), §31 (*Math*., V, 153-4); *Metaphysical Foundations of Mathematics* (*Math*., VII, 18-19).

[83] "Similarities can sometimes be recognized using magnitudes; thus figures are similar when their corresponding angles are equal, likewise when their corresponding sides are proportional.... On the other hand, magnitudes, in turn, are discovered using similarities, as when we find the magnitudes of angles from the similarity of figures, or the magnitudes of numbers from the identity of certain ratios. It sometimes also happens that something requires a very lengthy demonstration when investigated using magnitudes, but can be proved quite easily using similarities, as for example that equiangular triangles have homologous sides, or that a circle is proportional to the square of its diameter." *New Elements of Universal Mathematics* (LH IV 7B, 6 Bl. 9-10).

[84] *On the Analysis of Situation* (*Math*., V, 181-2); *Example of an Enlightening Geometry* (*Math*., VII, 281); *Metaphysical Foundations of Mathematics*: "From this it is clear that two equiangular triangles have proportional sides, and conversely" (*Math*., VII, 19). Cf. Leibniz to Tschirnhaus, May 1678: "that the sides of triangles having equal angles are proportional can be demonstrated by means of combinatorial theorems (concerning the similar and the dissimilar) far more naturally than Euclid did" (*Math*., IV, 460; *Brief.*, I, 380), and Leibniz to Arnauld, 12 July 1686: "I have many notable theorems of a geometrical form concerning causes and effects, also concerning similarity, and I give a definition of the latter from which I demonstrate easily several truths that Euclid proves in a very circuitous way" (*Phil*., II, 62). These considerations offer a striking analogy to those found in the *Prolégomènes philosophiques de la Géometrie* of Delbœuf, who most certainly was not aware of Leibniz's essays.

[85] "And generally it follows that a surface is similar to the square of its homologous sides, a body to the cube of its homologous side. From this Archimedes inferred that the center of gravity of similar figures is situated similarly (*Math*., VII, 276); cf. *On the Analysis of Situation* (*Math*., V, 182); *Metaphysical*

12. It is on this definition of similarity that there rest the definitions of the fundamental figures of geometry given in *On Euclid's Elements*. These include the definitions of a straight line ("A straight line is a line any part of which is similar to the whole") and of a plane ("A surface in which a part is similar to the whole").[86] Leibniz noted, in addition, that a solid, a plane, and a straight line are uniform in their interiors (or as we would say, homogeneous), so that two solids, two planes, or two straight lines that have the same extremities (or limits) coincide completely. He refined this idea even further by remarking that a circle and a helix are also uniform lines, and that a sphere and a cylinder are uniform surfaces, that is, all their parts are equal. But he added that their parts are not *similar*, as are the parts of a straight line and a plane, so that this property suffices to distinguish the latter two.[87]

In this same work we find definitions that derive from another train of thought and which would lead to a different system. They are no longer based on the idea of similarity but on the idea of symmetry. A plane is a section of a solid that has the same relation to its two sides; a straight line is a section of a plane that has the same relation to its two sides.[88] This is what is signified by saying that a straight line is *reversible* (in the plane) and that a plane is *reversible* (in space): this means that both of them coincide with themselves after being reversed. These, again, are ideas that modern geometers have rediscovered through an investigation of the principles of geometry.[89] However, they are not sufficiently primitive, for they involve the (metric) notions of congruence and motion.

13. In any case, Leibniz did not pursue his analysis of relations of similarity.[90] He seems to have given up on this method and to have tried to provide a basis for his new geometry using the relations of congruence and inclusion, on which he had already based his logical calculus and which are, as we have seen, common to logic and geometry. This, at any rate, was the intention he announced in an unpublished fragment, and that he began

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Foundations of Mathematics (*Math*., VII, 24). We note the analogy between this postulate of Archimedes and the principle of the balance (Chap. 6, §24); both of them derive, ultimately, from the principle of reason.

[86] Math., V, 185, 188. "Two similar lines cannot be contained in each other unless they are straight lines; thus the arc of a circle cannot be a part of another similar arc. Likewise, two similar surfaces cannot contain each other unless they are planes (LH IV 7B, 2 Bl. 54). Leibniz added that unlike lines and surfaces, two solids can contain each other, whether they are similar or not.

[87] These are precisely the ideas that Delbœuf has rediscovered in our time, and which he has expressed using the terms *isogenous* and *homogeneous*, defining a straight line as the only homogeneous line, and a plane as the only homogeneous surface (a circle and a sphere being only isogenous).

[88] *A Properly Geometrical Analysis*, §§11 and 13 (*Math*., V, 174). Cf. *On Euclid's Elements*, IV, 4; VII, 6: "A *straight line* is a section of a plane having the same relation to both sides. A *plane* is a section of a solid having the same relation to both sides" (*Math*., V, 185, 189).

[89] See, for example, Calinon, *Etudes sur la sphère, la ligne droite et le plan*, chap. III, §2, nos. 53-55 (Nancy and Paris, Berger-Levrault, 1888).

[90] By his own admission, however, this study did suggest to him the idea of his geometric calculus: "Furthermore, this consideration, which offers a very easy way of demonstrating truths that are difficult to prove any other way, also revealed to us a new kind of calculus, which is completely different from the algebraic calculus; it is similar to that calculus in its notation, but new in its use of that notation, or in its operations." *On the Analysis of Situation* (*Math*., V, 182).

to carry out.[91] He attempted even to define the continuous using only the idea of inclusion, without appealing to the idea of similarity, to any metric ideas, or to the idea of motion.[92] This definition of the continuous, moreover, is similar to that of Aristotle: It consisted in saying that an object A is continuous if, whenever A is decomposed into two objects B and C that together make up A,[93] these two objects have some common element (not a common *part*, but a common *boundary*).[94]

The same line of thinking led Leibniz to attempt to define figures by means of the idea of a *section*, which is at bottom the idea of an element common to two figures and appears independent of any notion of size or motion.[95] A line, for example, would be that figure such that every section induced by one of its points coincides with the point itself.[96] This definition, however, is neither general nor precise, for a section of a line may include an infinity of points other than the point in question, and even a continuous line segment. Leibniz noted, moreover, that the consideration of sections is equivalent to the consideration of motion, of which a section is merely a trace.[97]

He thus soon returned to the concept of motion, and defined lines, surfaces, and solids in terms of their being engendered by a displacement of points, lines, and surfaces, respectively. This definition rests on the idea of a path or trajectory (*tractus*), which Leibniz defined as a "successive continuous locus."[98] This definition , then, involved the idea of time,[99] and, what is more, the rather complicated notion of the continuous deformation of a figure during its displacement, which itself presupposes the notion of congruence. Thus, the consideration of motion at bottom comes back to that of congruence.

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[91] "I can, indeed, provide a foundation for a *new kind of geometry*, appealing only to the principle of inherence, i.e. only to *Epharmostiis*, like congruents, and not similarity or *Morphics* (LH XXXV, I, 14c).

[92] "Here I came upon some remarkable things: the idea of the continuous and of a part, and hence of the homogeneous, without relying on similarity, transformation, or motion" (*ibid*.).

[93] That is, such that A + B = C.

[94] "A is *continuous* if, whenever we take any two parts B and C that together exhaust A, they have some common D, or something existing in [*inexistens*] both B and C" (*ibid*.). Cf. *Example of an Enlightening Geometry*: "If there are three continuous things, X, Y, and Z, and every X is either Y or Z, and some X is Y, and some X is Z, then some X is both Y and Z" (*Math*., VII, 285). We note this logical expression of relations of inclusion.

[95] "A *section* of a magnitude is whatever is common to any two parts of the magnitude that do not have a common part." *On Euclid's Elements*, I, 3 (*Math*., V, 184). This definition, like that of a continuum, rests on the essential distinction between what is contained (*inexistens*) and a part (see Chap. 7, §9).

[96] "Here the general notion of a line, without any consideration of motion or surface, likewise the notion of width and depth.... A *line* is that extension such that any section through a given point is that point (LH XXXV, I, 14a).

[97] "Let us see whether it is easier to make use of motion than sections; for, after all, sections are the traces of something in motion. In this way, we will be no less able to refrain from any consideration of similarity, relying on the notion of congruence alone.... A *line* is that extended thing described by the motion of a point" (LH XXXV, I, 14b). Cf. *General Reflections on the Describing of Lines Through Motion* (LH XXXV, I, 18).

[98] *Geometrical Characteristic*, §§12-13 (*Math*., V, 145); *Metaphysical Foundations of Mathematics* (*Math*., VII, 20-21).

[99] Leibniz observed that we can define a line in terms of time and motion, as the locus of a point coordinated at successive instants (*On Euclid's Elements*, I, 2; *Math*., V, 183). However, it is important to note that this definition in no way implies the continuity of the line; the latter no more follows from the continuity of time than the continuity of a function follows from the continuity of the independent variable.

14. It is, therefore, on the notion of congruence, rather than the notions of similarity and motion, that Leibniz preferred to ground the fundamental definitions of geometry.[100] Congruence (i.e., geometric equality, or the possibility of coincidence) is, as we know, the union of the relations of similarity and equality (or quantitative equivalence).[101] Now, all points are essentially equal and similar, and hence congruent.[102] By the mere fact that they are parts of the same space, any two points stand in a certain relation of situation, which is their distance or separation from each other.[103] This relation must be able to remain constant when the points are displaced together. In order to depict this mutual displacement, Leibniz imagined that the two points are part of an arbitrary continuum, which gets displaced all in one piece. In short, he assumed the "axiom of congruence" or "free mobility," without which there is obviously no coincidence nor the condition for any possible measure.[104] We note that this notion of the constant distance between two points is independent of the notion of a straight line and prior to it.

Two pairs of points are always similar, since they are indiscernible when taken separately.[105] But they are not always congruent. In order for them to be congruent (supposing that each is part of a solid continuum), we must be able to make them coincide with one another, or else coincide with a pair of fixed points.[106] It is by means of similar congruences (and their combinations) that Leibniz tries to define all the elementary figures.

15. In what follows, we designate given or determinate points by the initial letters of the alphabet, and unknown or variable points by the final letters. A congruence containing a variable point determines, in general, a locus, namely the set of points which, when substituted for the variable point, make the congruence true. Thus the simplest congruence

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[100] "But here we shall employ just *congruence* in order to explain matters of situation, setting aside *similarity* and *motion* for another occasion." *A Purely Geometrical Analysis* (*Math*., V, 172).

[101] Leibniz employed different signs for congruence in different periods (see p. 311, n. 4). We employ throughout the modern sign for arithmetical congruence (≡).

[102] *Geometrical Characteristic*, §§9 and 10 (*Math*., V, 144); *A Purely Geometrical Analysis*, §9 (*Math*., V, 173-4); LH XXXV, I, 5d.

[103] *Geometrical Characteristic*, §11 (*Math*., V, 144-5).

[104] See Russell, *Essay on the Foundations of Geometry*. In the *Geometrical Characteristic* (§78-82) there are found several postulates that are merely forms or particular cases of the axiom of free mobility, namely: 1) that any figure can be moved in space; 2) that one of any two figures can be moved while the other remains at rest; 3) that any *path* can be moved in such a way that one of its points coincides with any given point, or 4) in such a way that one of its points remains fixed (*Math*., V, 164). Cf. other postulates at §§60-67 (*Math*., V, 161). Similarly, Leibniz defines a straight line there using the idea of motion, be it (§15) as the set of points of a solid that remain fixed as it is turned around two fixed points, or (§83) as a *path* that cannot be moved once two of its points are fixed (*Math*., V, 147, 164). We shall see that we obtain the same definition using the idea of congruence, so that the latter completely replaces the idea of motion.

[105] LH XXXV, I, 5d.

[106] "Two points (A and B) have the same situation with respect to each other as two other points C and D, if both pairs can be made to coincide with two points L and M of the same continuum.... In this case I say that the situation of points A and B is congruent to that of points C and D" (LH XXXV, I, 3e). Leibniz adds that if we now have the congruence AB ≡ CD, it is only later, once we have defined the straight line and its length, that we could write the *equality* (of magnitude), AB = CD.

$$A \equiv X$$

can be regarded as defining space, since, according to what has been said, every point in space is congruent to the given point, A.[107]

When the two members of a congruence are composed of several points, the congruence signifies that the two figures formed from these points can be made to coincide by making their corresponding points coincide simultaneously. Thus the congruence

$$AB \equiv CD$$

indicates that we can *at the same time* make A coincide with C and B with D; this means that the two pairs of points AB and CD are congruent, or that they have the same distance with respect to each other. Similarly, the congruence

$$ABC \equiv DEF$$

signifies that we can *at the same time* make A coincide with D, B with E, and C with F. It implies, further, that the three following congruences hold:

$$AB \equiv DE \quad BC \equiv EF \quad AC \equiv DF$$

These three simultaneous congruences, in turn, entail the preceding one, so it is equivalent to the three of them together.[108]

With this assumed, the congruence

$$AB \equiv AX$$

defines the locus of points whose distance from point A is the same as that of point B, or, in the words, the sphere with A as its center and radius AB (understanding by "radius" not the line AB but the distance between the two points A and B, which are invariably linked to each other).[109]

The congruence

$$ABC \equiv ABX$$

defines the locus of points whose distances from points A and B are the same as that of point C. This locus is *in general* the circumference of a circle, whose center is on the line AB and which itself lies in the plane perpendicular to this line (we say this in advance,

[107] Leibniz to Huygens (*Math*., II, 22; *Brief.*, I, 572); *Geometrical Characteristic*, §§68, 89, 90 (*Math*., V, 161, 166).

[108] Leibniz to Huygens (*Math*., II, 22, 24; *Brief.*, I, 572, 575); *Geometrical Characteristic*, §43 (*Math*., V, 157); *A Purely Geometrical Analysis*, §4 (*Math*., V, 173).

[109] Leibniz to Huygens (*Math*., II, 23; *Brief.*, I, 573); *Geometrical Characteristic*, §§88, 91, 92, 94 (*Math*., V, 165, 166).

for greater clarity). We can, therefore, take this congruence as the definition of a circle.[110] There is this very remarkable fact about it: it involves neither the idea of a straight line nor even that of a plane, nor does it presuppose a given or known center.[111]

16. This locus is in general, we have said, a type of curve; however, there is one exceptional case in which it is reduced to a point, namely, the point C. In this case, point C will be, by definition, *in the direction* AB; it will be, as Leibniz says, *unique in its situation* with respect to AB. As a result, we may define a straight line as the locus of points which are unique in their situation with respect to two given fixed points.[112] We see that motion is in no way essential to this definition and serves only to make it more intuitive: a straight line, then, is the locus of those points which remain stationary when a solid (or even the entire space) is rotated around two fixed points.

This definition implies a postulate, namely that there exist such points. Leibniz was well aware of this, and he tried to prove that given two points, one can always find a third situated on their direction; but his demonstrations do not appear to be valid, and the fact that he proposed several of them seems to indicate that he did not find them very convincing.[113]

But let us accept this point; from the preceding definition it follows that if three points A, B and C lie in a straight line, the congruence

$$ABC \equiv ABX$$

implies the identity (or coincidence) of the points C and X. Conversely, if this congruence necessarily implies the identity of these two points, then the three points, A, B and C lie in a straight line.

17. The best commentary on this definition can be found in the criticism Leibniz made on many occasions of the classic definition of Euclid.[114] Vitale Giordano had published, in 1686, *Euclide restituto*, in which he substituted for Euclid's definition that of Heron (one that was, according to him, clearer for beginners), which defines a straight line as the

[110] Leibniz to Huygens (*Math*., II, 23; *Brief*., I, 573); *Geometrical Characteristic*, §73 (*Math*., V, 162); cf. §§84, 96 (ibid., 165, 166). In the plane, a circle will be defined by the simplest congruence AB ≡ AX (*A Purely Geometrical Analysis*, §25; *Math*., V, 176).

[111] Leibniz to Huygens (*Math*., II, 24; *Brief*., I, 574).

[112] "A *straight line*... is the locus of all points unique in their situation with respect to two points." *On Euclid's Elements* (*Math*., V, 185). In an unpublished fragment (LH XXXV, I, 1a), Leibniz says less precisely, "A straight line is unique in its kind between its two end points." He also alludes there to a definition of Jungius.

[113] *Geometrical Characteristic*, §51(*Math*., V, 159). In *Demonstration of Euclid's Axioms*, 22 February 1679 (LH XXXV, I, 2), Leibniz defines a straight line as that line uniquely and entirely determined by two points. He then wonders whether there exists such a line, and thinks that he can prove that there does by invoking this notable axiom: "from any two things taken at the same time, something new is always determined, for it is something more to suppose them at the same time than to suppose them one at a time." Cf. *Geometrical Characteristic*, 10 August 1679, §11 (*Math*., V, 144). It is by very similar considerations that Mr. Russell attempts to establish the axiom of the straight line (*Essay on the Foundations of Geometry*, §138).

[114] It is well known that this definition (which runs literally: "a straight line is that line which rests equally on its points") has, because of its obscurity, given rise to numerous interpretations. See Leibniz's commentary on it in his *Geometrical Characteristic*, §75 (*Math*., V, 164; cf. n. 126).

shortest path between two of its points.[115] Leibniz objected to him that the majority of theorems concerning straight lines invoked neither this definition nor that of Euclid, an indication that they were useless and even wrong; for what good is a definition if it does not enter into demonstrations? We do not know, then, which line we are talking about, or whether the theorems indeed concern the same line as the definition.[116] Leibniz subsequently proposed defining a straight line as the locus of points that remain stationary in the rotation of a body; or again, as the line that divides a plane into two congruent parts (and similarly, a plane as the surface that divides space into into two congruent parts). Giordano objected to him that these definitions of a straight line presupposed the notions of a (solid) body and a plane. Leibniz replied that the notions of body and plane were, indeed, for him anterior to that of a straight line, and that the simplest and most primitive definitions of a plane and a straight line were those which represented them as sections or intersections.[117]

The same criticism of Euclid is found again in the *New Essays*:[118] "Euclid's definition is obscure, and plays no role in the demonstrations.... Lacking a distinctly expressed idea, i.e., a definition, of a straight line," Euclid was obliged to make use of two axioms:1) two straight lines have no common segment, and 2) two straight lines do not enclose a space.[119] With regard to the last axiom, which he took to be purely intuitive, Leibniz expressed these important thoughts on the geometric method: "The imagination, derived from sense experience, does not allow us to represent more than one coincidence between two lines, but this is not what a science should be founded on." We should not believe that "the imagination supplies the connection between distinct ideas.... These sorts of images are merely confused ideas, and whoever comprehends a straight line by this means alone will not be able to prove anything about it."[120] Thus, in geometry as in all things, Leibniz wanted (and believed himself able) to return to clear and distinct ideas, which alone are primitive and simple, and to demonstrate all the axioms suggested by intuition by reducing them to true definitions.[121]

This, at any rate, is what he accomplished for the notion of a straight line (assuming the existence postulate noted above). From the fact that a straight line is determined by two of its points, he deduced immediately that two lines cannot have a common segment without coinciding entirely, that they cannot enclose a space, and in short all the properties that Euclid attributed to straight lines and which comprised for him so many axioms.[122]

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[115] Leibniz from Vitale Giordano (*Math*., IV, 198).

[116] Leibnizs to Vitale Giordano, 1689-90 (*Math*., IV, 196, 199). Cf. *On Euclid's* Elements, IV, 1 (*Math*., V, 185).

[117] *Math*., IV, 198, 199.

[118] *New Essays*, IV.xii.6; see also, earlier, *Animadversions Concerning the Principles of Descartes* (1692), where Leibniz says that Euclid would have been able to prove the axiom of the straight line if he had had an adequate definition of it (*Phil*., IV, 355). Cf. p. 199, n. 2

[119] *On the Analysis of Situation* (*Math*., V, 179).

[120] Leibniz also criticizes Archimedes, who "gave a type of definition of a straight line, saying that it is the shortest line between two points." This is Heron's definition, adopted by Giordano.

[121] Cf. Chap. 6, §§12 and 13. In *Demonstration of Euclid's Axioms* (22 February 1679), he maintained that the geometrical calculus should be grounded in ideas and not in the senses or the imagination (LH XXXV, I, 2). Thus he believed that one could and should set out the elements of geometry in a different way from Euclid. Leibniz to Hermann, 11 May 1708 (*Math*., IV, 328).

[122] *A Purely Geometrical Analysis*, §§20-23 (*Math*., V, 176).

18. We may now turn to the plane. This Leibniz defines as the locus of points equidistant from two given points (in space). Let A and B be two such points; the plane is represented by the congruence

$$AX \equiv BX$$

We know from classical geometry that this plane is the plane perpendicular to the line AB through its midpoint, but this notion plays no role in the preceding definition. We have, in short, defined the straight line, plane, circle, and sphere in terms of congruences that involve only the notion of the (invariable) distance between two points, and each of these definitions is independent of the rest. In one sense this is highly advantageous; in another, it is a serious inconvenience, for we now need to reestablish relations of situation among these fundamental figures, and this is not always easy. To do this, Leibniz undertook to determine the intersections of these various figures with one another.

We have already proved that two straight lines can have only one point in common, so that their intersection (if it exists) is a point. We can also show that two spheres, or a sphere and a plane, have a circumference as their intersection.[123] In the same way, we establish that the circumference is situated on a sphere and in a plane. This last property, however, is proved only for those circumferences whose points are equidistant from their two poles (A and B), and not for any arbitrary circumference ABC ≡ ABX. This exception was noted by Leibniz himself.[124]

19. It remains to show that the intersection of two planes is a straight line, and that a straight line is entirely contained in a plane. For this, Leibniz appealed to a different definition. Given three points, A, B, and C, that do not form a straight line,[125] the locus of points equidistant from these three points is a straight line. The latter is thus defined by the double congruence

$$AX \equiv BX \equiv CX$$

[123] Here is how: Suppose A and B are the centers of two spheres and C is a point common to them (assuming that they have a common point). These spheres will be represented by the two congruences AC ≡ AX and BC ≡ BY. We find the locus of points common to the two spheres by replacing Y with X. This gives AC ≡ AX and BC ≡ BX; hence ABC ≡ ABX is a congruence representing a circumference. On the other hand, consider the plane represented by the congruence AX ≡ BX and the sphere represented by the congruence AC ≡ BY. The locus of their common points is defined by the union of the two congruences AX ≡ BX ≡ AC. Suppose now that the point C is common to the sphere and the plane. Then we have AC ≡ BC; hence also, BC ≡ BX and AC ≡ AX, from which we get ABC ≡ ABX, a congruence representing a circumference. Leibniz to Huygens (*Math*., II, 24; *Brief*., I, 574-5); *Geometrical Characteristic*, §103 (*Math*., V, 168).

[124] *Geometrical Characteristic*, §86 (*Math.,* V, 165).

[125] This restriction would amount to a vicious circle, if it did not presuppose the other definition of a straight line.

This is the straight line perpendicular to the plane ABC, at the center of the circle inscribed in the triangle ABC.[126]

What is the relationship of this definition to the first, and how does Leibniz derive the one from the other? He gave absolutely no indication of this. In the *Geometrical Characteristic* (§75), he attempted to show that the points X, so defined, form a straight line and are situated in the same direction. He even commented in this context on the definition of Euclid,[127] which would seem, rather, to be related to the first definition. In any case, he did not prove the equivalence of his two definitions; in this regard, he tangled himself up in confused considerations, which he appears only to have been able to overcome by going back to basics, as if he were setting himself straight,[128] and returning soon (§83) to his first definition of a straight line.

Be this as it may, if we admit the second definition, it is easy to demonstrate that the intersection of two planes is a straight line. Let the two planes be defined by the congruences

$$AX \equiv BX \quad BY \equiv CY$$

To find their intersection, we identify X and Y; this gives us

$$AX \equiv BX \quad BX \equiv CX$$

Together, these two congruences define a straight a line.

We can show in the same way that the intersection of two straight lines is a point. Let the two lines be

$$AX \equiv BX \equiv CX \qquad BY \equiv CY \equiv DY$$

To find their intersection, we again identify X and Y; this give us

$$AX \equiv BX \equiv CX \equiv DX$$

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[126] This definition is the only one given in the letter to Huygens (*Math*., II, 24; *Brief*., I, 574). It is found again in the *Geometrical Characteristic* (§§75, 87, 97), subordinated to the first definition (*Math*., V, 163, 165, 167). There Leibniz introduced a supplementary condition AB ≡ BC ≡ AC, which is unnecessary, but which was suggested to him by the less simple construction of §74, in which a circle figures. In the *Properly Geometrical Analysis*, where beginning in §16 Leibniz restricts himself to the plane, he defined a straight line as the locus of points having the same relation with respect to two other points (that is, that are equidistant from these two points). We know that this locus is a straight line perpendicular to the midpoint of the segment joining the two points (§26; *Math*., V, 176). A straight line is then represented by the congruence AX≡BX, which has the same form as that representing a plane in space. It is, in fact, the intersection of the plane in question with the plane AX ≡ BX (§28; *Math*,. V, 177).

[127] "And so we see what Euclid meant when he said that a straight lies equally between its points, that is, does not go up and down in any part, or is not related by a lasting motion in any different way to point A than to points B or C" (*Math*., V, 164).

[128] "Let us reconsider certain things." *Geometrical Characteristic*, §76 (*Math*., V, 164).

There is only a single point equidistant from four given points. The locus is, therefore, reduced to a point.[129]

The same uncertainty, or ambiguity, is met again in plane geometry. Sometimes Leibniz defined a straight line as the locus of point having a unique relation with respect to two given points;[130] sometimes he defined it as the locus of points having the same relation with respect to two given points, that is, which are equidistant from them.[131] The first definition allowed him to establish the characteristic properties of a straight line (§§20-23). The second allowed him to prove that two circles intersect at only two points, since a straight line cannot have the same relation to three points in a plane. It follows that a straight line and a circle can have only two points in common, and that a circle is determined by three points. However, there remains an inconsistency between the two definitions of a straight line, which are invoked one after another.

Leibniz did try to unify the two definitions by setting forth the following proposition. If we have

$$DABC \equiv EABC \equiv FABC$$

with D, E, and F nonidentical, the three points A, B, and C lie in the same direction.[132] But he seemed to assume it as a definition,[133] even though he had already defined *direction* in another way (§50). This would, therefore, be rather a theorem, but it is not demonstrated, so it actually amounts to a third definition of a straight line, which is independent of the other two or else a more complicated form of the second.[134] In sum, Leibniz involved himself in difficulties and inconsistencies; which he does not appear to have overcome; he did not succeed in establishing his geometrical calculus on clear and consistent principles.

20. It is instructive to investigate the reasons for his failure—not the accidental reasons we have just indicated, but the deep and general reasons that reside in the principles of the system. For this, it is only necessary to ask whether Leibniz actually did what he

[129] We read in §99 of the *Geometrical Characteristic*: "If we let AY ≡ BY ≡ CY, the locus Y will be a *point*, or Y will be satisfied only uniquely... This proposition must be demonstrated." There is an apparent contradiction here with §97, where Leibniz writes: ""If we let AY ≡ BY ≡ CY, then the locus of all Y will be a *straight line*" (*Math*., V, 167). This is explained if we suppose that in this passage Leibniz, without saying so, is restricting himself to the plane.

[130] *A Properly Geometrical Calculus*, §§18-19.

[131] *Ibid*., §26.

[132] *Geometrical Characteristic*, §57. Leibniz added that the three points D, E, F lie in the same plane and on the same circle, which is obvious.

[133] "Any three points A, B, C will be said to lie on a straight line..." (*Math*., V, 160).

[134] It would still have to be established that a straight line is entirely contained in any plane that contains two of its points. This is what Leibniz did not do and what would not easy with the preceding definitions. On the other hand, it is easy when one gives a definition of the plane analogous to that of a straight line: "the locus of all those points unique in their situation with respect to three points not falling on the same straight line" (*On Euclid's Elements*; *Math*., V, 189). It is still possible with two related definitions of a plane as an reversible surface, and of a straight line as a reversible line, or as a symmetrical section of a plane. However, it would be necessary to opt definitively for one set of definitions, whereas Leibniz remained undecided and drifted between the different systems that he proposed one after another.

wanted to do, that is, whether he freed geometry from any consideration of magnitude and managed to express situation directly.

The answer to this question can only be, no. We have seen that Leibniz neglected relations of similarity in order to study almost exclusively the relation of congruence, the least general and most complex of all the geometrical relations. Moreover, this relation in no way abstracts from magnitude, since it involves, along with the relation of similarity, that of quantitative equality. Finally, the method employed by Leibniz does not give *intrinsic* definitions of figures, as would be necessary for it to be able to surpass analytic geoemtry in simplicity and intuitive clarity. No less than the latter, it requires external reference points for figures and auxiliary external relations, in short, a system of coordinates.[135]

In fact, Leibniz's geometrical calculus reduces to analytic geometry, for it is essentially nothing more than a coordinate system (two-dimensional in the plane, three-dimensional in space), in which a point is defined by its distance with respect to two or three fixed reference points. Not only is it an analytic system that expresses (like Descartes's system) situation by means of relations of magnitude, but it is a less convenient and less satisfactory system, on account of the ambiguity of its determinations.[136]

To establish the properly geometric calculus of which he dreamed, Leibniz would have had, on the contrary, to separate relations of situation from relations of magnitude, and to abstract from any *metrical* consideration. It would not have been enough to substitute similarity for congruence as the fundamental relation, for as he showed,[137] similarity still involves a relation of magnitude—namely proportionality. It would have been necessary to complete the analysis of situation by reducing figures to projective relations and properties.[138] From this point of view, however, the only primitive relation

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[135] For example, a plane is defined by means of two arbitrary external points; how would one recognize the identity of a plane defined by two pairs of points? It is the same for a circle and a straight line, at least if one defines it as the locus of points equidistant from three given points. As to the first definition, it is undoubtedly *intrinsic*, but it does not amount to a general formula or equation for a straight line and it does not provide, as the latter does, a means of constructing it.

[136] To each set of bipolar coordinates there corresponds two points symmetrical in relation to the straight line or plane that contains the reference points (or poles). Leibniz was deceived by this ambiguity; in effect, he confounded in his calculations symmetrical tetrahedrons with congruent tetrahedrons. He wrote, for example, ABCY ≡ ABDY, to express the fact that the points C and D are symmetrical with respect to the plan ABY, because the following hold separately: AC ≡ AD, BC ≡ BD, CY ≡ DY (*Geomtrical Characteristic*, §98). We see from this that his notation did not allow him to distinguish congruent figures from symmetrical figures, whose homologous parts alone are congruent. This shows clearly that it did not take account of the relative *situation* of these parts and was incapable of expressing it. For this, it would have been necessary to ascribe a direction, and thus a sign (positive or negative), to the relevant line segment, triangle, or tetrahedron, whereas Leibniz, considering only magnitude, assumed on the contrary as an axiom AB ≡ BA (ibid., §42). Cf. p. 315, n. 3.

[137] See p. 412, n. 3.

[138] In order to show the different degrees of generality of the concepts of congruence, similarity, and projectivity, it is enough to say that, from the point of view of metric geometry, two pairs of points can differ in their respective distances and are equal only if their distances are equal. From the point of view of similarity, any two pairs of points are similar, but two sets of three points lying on a straight line may or may not be similar, depending on whether their distances are or are not proportional. Finally, from the point of view of projective geometry, any two sets of three points lying on a straight line are (projectively)

among points consists in their being part of the same straight line or the same plane, for this, as Leibniz recognized, is a unique situation with respect to these points, a determination of each of them in relation to the others. The fundamental projective operations are those of *projection* and *section*. Two figures are projectively equivalent, if they can be put in perspective, or if one can transform into the other through a series of projections and sections (alignments and intersections). If Leibniz had pursued this course, he would have founded the true geometry of position that Staudt established in the nineteenth century as a complete and independent system.[139]

On the other hand, in order to invent a geometrical calculus that would, as he desired, take for its elements points, rather than magnitudes, it would have sufficed to observe that two points determine a straight line, and three points a plane, and thus to regard a straight line and plane as *products* of the point that determine them. Conversely, a straight line as the intersection of two planes and a point as the intersection of two straight lines or three planes can be regarded as products of the planes or lines that determine them. We could thus express the operations of projection and section by a type of multiplication, which would have different properties and laws from those of arithmetical multiplication and which would serve as the basis of a new algebra.

These are are precisely the principles of the calculus of extension developed by Grassmann.[140] We can therefore regard it as the fulfillment of the geometrical analysis sketched by Leibniz, and this all the more as Grassmann, who had no knowledge of Leibniz when he discovered his calculus, was later led to present it as the realization of the latter's plan.[141] This marvelous convergence undoubtedly bestows honor on the genius of Grassmann, but it perhaps bestows even more on that of Leibniz, for it proves that his idea of a geometrical calculus was neither a fantasy nor trivial, as so many philosophers and mathematicians have believed. Like Boole, Grassmann rediscovered, or revived, a part of the universal characteristic; both confirmed the boldest of Leibniz's conceptions, by showing that they were not dreams but prophetic intuitions, which anticipated by nearly two centuries the progress of science and of the human mind.[142]

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equivalent, for they can always be put in perspective and only sets of (at least) four points can be distinguished from each other, according to whether or not they are projective.

[139] G.K. Christian von Staudt, *Geometrie der Lage* (Nuremberg, 1847). It is appropriate to recall that the fundamental ideas of projective geometry are already found implicitly in the work of Desargues (1593-1662).

[140] Hermann Grassmann, *Die Ausdehnungslehre* (1844), in *Gesammelte Werke*, vol. 1, part 1 (Leipzig, Teubner, 1894).

[141] See Appendix V.

[142] See the conclusion of our article "L'Algebre universelle de M. Whitehead," *Revue de Métaphysique et de la Morale*, vol. 8, p. 362 (May 1900).