Leibniz on Infinitesimals and the Reality of Force

Donald Rutherford (University of California, San Diego)

Leibniz's efforts to apply his differential calculus to the analysis of physical phenomena constitute one of the most forward looking aspects of his natural philosophy. Concealed in these efforts, however, are significant problems about the interpretation of the calculus and of his new science of dynamics. These problems come together in Leibniz's conception of force as a momentary endeavor that is represented in the calculus as an infinitesimal quantity. The idea of force as an infinitesimal element of action that is responsible for continuous changes in a body's state of motion has an undeniable intuitive appeal. Nevertheless, Leibniz articulates other views that make it difficult to see how such a conception of force can be defended.

According to Leibniz's dynamics, which he develops in opposition to Descartes's geometrical physics, active and passive forces are the only real properties of matter. As he writes in the programmatic *Specimen dynamicum* (1695): "Nihilque adeo in ipso reale est, quam momentaneum illud quod in vi ad mutationem nitente constitui debet. Huc igitur redit quicquid est in natura corporea praeter Geometriae objectum seu extensionem."[1] (GM VI, 235). Since for Leibniz the object of geometry—spatial extension—is merely "ideal," the real properties of matter are limited to its active and passive forces: its tendencies to initiate and to resist change. In the case of active force, Leibniz further insists that this force is something "momentary," by which he appears to mean that it lacks any finite duration.

---

[1] "there is nothing real in motion but a momentary something which must consist in a force striving toward change. Whatever there is in corporeal nature over and above the object of geometry or extension reduces to this" (AG 118). See also the unpublished part II of the *Specimen Dynamicum* (GM VI, 247/AG 130). Editions of Leibniz's writings are cited according to the list of abbreviations. Where a translation is cited, I have made use of it, though I have sometimes taken the liberty of modifying it slightly; where none is cited, the translation is my own.

For this reason, it is tempting to think of the basic elements of force as infinitesimal quantities: either infinitesimal amounts of endeavor that are summed to produce finite forces, or force states of infinitesimal duration—or both. Yet although Leibniz himself gives currency to this conception of force, in his reflections on the calculus he also expresses strong reservations about the coherence of the idea of an infinitesimal magnitude. To think of the differential *dx* as referring to a quantity smaller than any finite quantity, he says, is to operate with a "fiction," which is useful for calculating but does not designate any real entity.[2] This is because, strictly speaking, there cannot be such an entity: an actual amount, length or duration that is smaller than any finite amount, length or duration.

Whether Leibniz is correct in this judgment is open to debate.[3] Clearly, however, it raises significant concerns for his theory of force. If, as he believes, infinitesimal quantities of distance or speed are "fictions," are we not obliged to say the same about the momentary forces to which he appeals in explaining continuous change? If we are, then we are faced with an apparent inconsistency at the heart of his dynamics. Far from being the only real properties of matter, momentary forces turn out to be mere fictions. While an instrumentalism that counted all physical properties as fictions invoked to save the phenomena has its attractions, it is at odds with Leibniz's convictions concerning the reality of force. On the face of it, then, there appears to be no way for him to combine his preferred interpretation of the calculus with his understanding of physical force: if force is real, it cannot be an infinitesimal quantity; if it is an infinitesimal quantity, it cannot be real.

---

[2] "The infinitesimal calculus is useful with respect to the application of mathematics to physics; however, that is not how I claim to account for the nature of things. For I consider infinitesimal quantities to be useful fictions" (GP VI, 629/AG 230).

[3] For a modern defense of the mathematical coherence of infinitesimals, see Bell, 1998.

In some passages, Leibniz suggests that the appeal to infinitesimal forces should be viewed merely as a heuristic and not as an attempt to describe the nature of physical reality. Nevertheless, he explicitly claims that force is something "momentary," and so the question remains of how to conceive of force in a way that is consistent both with the role assigned to it in the explanation of continuous change and with his fictionalism concerning infinitesimals. In what follows I offer an analysis of Leibniz's position that locates the central problem in the conception of force as an infinitesimal quantity that sums over time or space to produce finite changes in a body's state of motion, or finite quantities of force. Such a picture of infinitesimal forces as the underlying causes of physical processes is at best a heuristic, which can lead us astray if taken literally. I also argue, however, that a conception of force as a determinate finite quantity, while adequate for the purposes of physics, gets us no closer to what force really is. In the end, I propose, it is Leibniz's view that, in and of itself, force is not a mathematically representable property. To understand the sense in which force is real, we must turn to a different theoretical framework altogether, that of metaphysics, wherein force is represented as a modification of a substantial power, or principle of change. From this perspective, force is something momentary for Leibniz, because at any assignable moment each substance has a determinate tendency to change, yet this tendency itself lacks any measurable duration.

1. Representing Continuous Change

An anonymous essay published in the inaugural volume of the proceedings of the Berlin Academy of Science (1710) offers the following description of Leibniz's new differential calculus:

> Hic dx significat elementum, id est incrementum vel decrementum (momentaneum) ipsius quantitatis x (continue) crescentis. Vocatur et differentia, nempe inter duas proximas x

> elementariter (seu inassignabiliter) differentes, dum una fit ex altera (momentanea) crescente vel decrescente; […] Porro ddx est elementum elementi seu *differentia differentiarum*, nam ipsa quantitas dx non semper constans est, sed plerumque rursus (continue) crescit aut decrescit. Et similiter procedi potest ad dddx seu d^3x, et ita porro […].[4] (*Monitum de Characteribus Algebraicis*, *Miscellanea Berolinensia*, vol. 1, 159-60 = GM VII, 222-3)

The account of the calculus presented here lends itself to an interpretation in terms of infinitesimal quantities. A continuous change in a finite quantity *x* is conceived to occur through the addition or subtraction of an element symbolized by *dx*, which is the difference between two minimally different values of *x*. For the change in *x* to be continuous, *dx* must be smaller than any finite difference; hence it is an infinitesimal quantity, added to or subtracted from *x* in a minimal interval of time (a "moment"). The author of the text further raises the possibility of a series of higher-order infinitesimals, representing inassignably small differences in the values of *dx*, of *ddx*, and so on. Thus, we appear to have a completely general interpretation of the calculus as a symbolism for representing continuous changes in both finite magnitudes and infinitely small magnitudes, in terms of the addition or subtraction of infinitesimal elements.

In his writings on dynamics Leibniz draws frequently on this interpretation of the calculus. His most prominent application of infinitesimals to the analysis of motion is found in the *Specimen dynamicum*. There he begins by distinguishing two senses of motion: the continuous path traveled by a body in a finite interval of time (motus), and the motion of a body in an instant (Motio):

> Quin etiam quemadmodum (non incommode ad usum loquendi doctrinalem) ab accessu jam facto faciendove distinguere possumus accessionem quae nunc fit, tamquam incrementum accessus vel elementum; aut quemadmodum descensionem praesentem a facto jam descensu, quem auget,

[4] "Here *dx* signifies an element, that is, a (momentary) increment or decrement of the (continuously) increasing quantity *x*. It is also called a *difference*, namely that between two minimally (or inassignably) different proximal [values of] *x*, where one arises from the other that is (momentarily) increasing or decreasing […]. Furthermore, *ddx* is an element of an element, or *a difference of differences*, for the quantity *dx* itself is not always constant, but often in turn (continuously) increases or decreases. And similarly, one can proceed to *dddx* or d^3x, and so on […]." - For the ascription of this text to Leibniz see Cajori, 1928-29, 2, 195.

> distinguere licet; ita possemus praesentaneum seu instantaneum motus elementum ab ipso motu per temporis tractum diffuse discernere et appellare Motionem.[5] (GM VI, 237)

Again, a continuous change in a finite quantity, a body's motion through space, is explained in terms of the addition of "instantaneous elements of motion." The summation of these elements (*ds*) over time gives the elapsed motion, or path (*s*). Leibniz's stated purpose in distinguishing these two senses of motion is to clarify the significance of Descartes's measure of force as "quantity of motion," defined as the product of a body's size and speed.[6] Since *ds* represents the distance traveled by a body in an instant, it is proportional to the body's instantaneous speed. Consequently, the Cartesian quantity of motion is properly understood as a body's "momentary quantity of motion," which Leibniz identifies as its *impetus*. Quantity of motion itself, he argues, is more accurately explained as the quantity that "ex aggregatu impetuum durante tempore in mobili existentium (aequalium inaequaliumve) in tempus ordinatim ductorum nascatur."[7] (GM VI, 237).

Leibniz thus proposes to explain the generation of finite continuous quantities—a body's extended motion or its quantity of motion—as sums of infinitesimal quantities. Motion through space is generated by the successive addition of instantaneous elements of motion, and a body's quantity of motion is explained as the sum over time of its momentary quantities of motion. In

---

5 "To speak in a way not inappropriate for scientific use, just as we can distinguish the progress we are now making from the progress we have made or will make, considering our present progress as an increment or element of progress, or just as we can distinguish the present descent from descent already made, descent which it augments, so too we can distinguish the present or instantaneous element of motion from that same motion extended through a period of time, and call the former *motio*" (AG 120).

6 Cf. Descartes, *Principia Philosophiae*, II, 36 and 43 (AT VIII-1, 61 and 66-7). Although Leibniz routinely uses the term "velocitas" to refer to a body's speed, he recognizes the difference between speed and "directional speed" ("celeritas respectiva"), or velocity, and draws on it in his reformulation of the laws of motion; see GM VI, 493-94, and Garber, 1995, 314-19. In the present context the distinction is unimportant. (In the *Specimen dynamicum* Leibniz further complicates matters by labeling the modern notion of velocity "conatus": "However, just as a mobile thing existing in motion has motion [motum] in time, so too at any moment it has *speed* [*velocitatem*], which is greater to the extent that more space is traversed in less time. Speed taken together with direction is called *conatus*" (GM VI, 237/AG 120).)

7 "arises from the sum of the impetuses (equal or unequal) existing in a moving thing during a time, multiplied by the corresponding time" (AG 120).

cases of uniform motion, Leibniz recognizes that neither of these analyses is strictly necessary. If a body of mass *m* moves with constant speed *v*, its path over the interval *t* is given as the product *vt*, and its quantity of motion as *mvt*. Thus, the mathematical analysis of uniform motion can proceed independently of any positing of infinitesimal quantities. In Leibniz's view, however, non-uniform motion, in which a body's speed increases or decreases with respect to time, must be treated differently. In his unpublished *Dynamica* (1690), he appeals to the differential calculus as a means of representing the relevant changes in a body's state of motion:

> Quamdiu [...] velocitates mobilis eaedem per quasvis temporis partes (motu existente uniformi), sufficit calculus praecedens per quantitates vulgo receptas. Sed si variet ubique [...] velocitas in loco aut tempore, ad quantitates numero infinitas et magnitudine infinite parvas veniendum est seu ad incrementa aut decrementa vel differentias duarum quantitatum ordinariarum proximarum inter se. Exempli gratia: Dum grave motum accelerat, duae proximae sibi velocitates *v* et *(v)* a me dicentur habere differentiam infinite parvam *dv*, quae est incrementum velocitatis momentaneum, quo transit mobile a velocitate *v* ad *(v)*. Itaque in Geometriam introduxi novum circa analysin infinitorum calculi genus, suo quodam Algorithmo alibi a me explicato instructum, ubi notis differentiae et summae eodem fere modo utor, quo notis radicis et potestatis in Algebra uti solemus.[8] (GM VI, 426-27)

Leibniz's insistence on the need for the calculus in the analysis of non-uniform motion may seem at first glance unmotivated. In free fall, for example, where a body's speed increases by a constant factor with respect to time, an analysis of its motion in terms of "quantitates vulgo receptas [commonly received quantities]" suffices. To be charitable to Leibniz, we might read him as anticipating a more general treatment of motion, in which rates of change of speed (or the infinitesimal increments *dv*) need not be constant. By drawing on the calculus, he can represent

---

[8] "So long as the […] speeds of the moving thing (existing with uniform motion) are the same throughout any parts of time, the preceding calculus by means of commonly received quantities suffices. But if […] the speed varies everywhere in place or time, we must turn to quantities infinite in number and infinitely small in magnitude, that is, to increments and decrements, or differences of two ordinary quantities proximate with respect to each other. For example, when a heavy object accelerates in motion, two speeds *v* and *(v)* proximate to each other are said by me to have an infinitely small difference *dv*, which is a momentary increase of speed, by which the moving thing passes from speed *v* to *(v)*. And so in geometry I have introduced a new kind of calculus concerning the analysis of infinites, laid out with its algorithm explained by me elsewhere, where I use signs for differences and sums in almost the same way as we are accustomed to use signs for roots and powers in algebra."

the path of any body with respect to time, provided only that changes in its state of motion are continuous, or occur through infinitely small increments or decrements of speed.

Although this obviously is one of the signal achievements of the differential calculus, Leibniz's preference for an explanation of non-uniform motion in terms of infinitesimals can be traced to specific assumptions of his dynamics. With respect to free fall, Leibniz denies that Galileo's rule that a body gains equal increments of speed in equal times should be admitted as a genuine law of nature.[9] This is because he believes that a body's natural motion—the motion proper to it—is uniform and rectilinear.[10] Consequently, any variation in a body's speed or direction of motion must be ascribed to the action of forces that effect a change in it. His account of these forces is far from transparent. In fact, we find two diametrically opposed explanations of how corporeal forces effect a change in a body's state of motion. In line with ordinary ways of thinking, Leibniz often represents these forces as ones that act mechanically through the impact of bodies. This is the basis of his vortex theory of planetary motion, presented in the *Tentamen de Motuum Coelestium Causis* (1689), which he defends against the rival theory of Newton's *Principia*. According to Leibniz, the orbital motion of celestial bodies is to be ascribed not to the

---

[9] In a letter to Varignon of October 10, 1706, he writes: "The simpler way is that which does not make acceleration foundational, when there is no need to do so. I have made use of this for more than 30 years" (GM IV, 151). See also GM VI, 453-54.

[10] *Dynamica*, part I, sec. 2, ch. 5, props. 1-2 (GM VI, 342) and part II, sec. 3, prop. 17 (GM VI, 502). The argument offered in Part II of the *Specimen dynamicum* for the rectilinear character of motion is premised on a claim about the nature of force: "since only force and the nisus arising from it exist at any moment (for motion never really exists, as we discussed above), and since every nisus tends in a straight line, it follows that *all motion is either rectilinear or composed of rectilinear motions*. From this it […] follows that what moves in a curved path always tries [conari] to proceed in a straight line tangent to it" (GM VI, 252/AG 135). Leibniz's reasoning parallels that of Descartes in *Principia Philosophiae*, II, 39, though he rejects Descartes's grounding of rectilinear motion in "the immutability and simplicity of the operation by which God preserves motion in matter" (AT VIII-1, 63). For God's constant activity, Leibniz substitutes a tendency grounded in a body's inherent force.

action of a *sui generis* gravitational force ("action at a distance"), but to the impact of the moving particles of an aetherial fluid.[11]

Yet this is not Leibniz's deepest account of the nature and action of physical forces. Although we commonly explain changes in a body's state of motion by appeal to the action of external forces, in the strictest sense, Leibniz claims, there is no real causal interaction among things—no case in which one thing acts directly on another by transferring motion or force to it. Therefore, any change in a body's state of motion must be ascribed to the action of *internal* forces.[12] In general, the forces that explain the changes that occur in bodies in collision are elastic forces proper to each: "*Corpora non agunt immediate in se invicem motibus suis, nec immediate moventur, nisi per sua Elastra.*"[13] (GM VI, 492).

For our purposes, it is unnecessary to negotiate between these two accounts of force, since the salient point applies to both: On the assumption that all change is continuous change, Leibniz maintains that it must be explained in terms of the action of infinitesimal forces. He refers to these forces generically as "dead force," indicating that their presence in a body does not depend upon its already being in a state of motion. In the *Specimen dynamicum*, Leibniz

---

11 "[I]t can first of all be demonstrated that according to the laws of nature *all bodies which describe a curved line in a fluid are driven by the motion of the fluid*. For all bodies describing a curve endeavor to recede from it along the tangent (because of the nature of motion), it is therefore necessary that something should constrain them. There is, however, nothing contiguous except for the fluid (by hypothesis), and no conatus is constrained except by something contiguous in motion (because of the nature of the body), therefore it is necessary that the fluid itself be in motion" (GM VI, 149; trans. in Bertoloni Meli, 1993, 128-29). See also *Specimen dynamicum*, Part II: "For if we assume something we call solid is rotating around its center, its parts will try to fly off on the tangent; indeed, they will actually begin to fly off. But since this mutual separation disturbs the motion of the surrounding bodies, they are repelled back, that is, thrust back together again, as if the center contained a magnetic force for attracting them, or as if the parts themselves contained a centripetal force. Thus, the rotation arises from the composition of the rectilinear nisus for receding on the tangent and centripetal conatus among the parts" (GM VI, 252/AG 135-36). For discussion of the details of Leibniz's theory, see Aiton, 1984, and Bertoloni Meli, 1993.

12 "Rigorously speaking, no force is transferred from one body to another, but every body is moved by an innate force [insita vi]" (A VI, 4, 1630/DLC, 333). See also A VI, 4, 1620; GP II, 195; and *Specimen dynamicum*, Part II: "every passion of a body is spontaneous, that is, arises from an internal force, even if it is on the occasion of something external" (GM VI, 251/AG 134).

13 "*Bodies do not act immediately on one another through their motions, nor are they immediately moved except through their own elasticity.*" - On the role ascribed to elastic forces, see Breger, 1984; Garber, 1995, 321-25.

locates dead force within a complex taxonomy of forces. Within the category of force in general, he distinguishes, on the one hand, active and passive force, and on the other, primitive and derivative force. For the moment, we are interested only in the class of active derivative forces: the physical forces by which bodies act, and to which the laws of motion apply. Among these forces, Leibniz is primarily concerned to emphasize the difference between "dead force" (*vis mortua*) and "living force" (*vis viva*): "Hinc Vis quoque duplex: alia elementaris, quam et *mortuam* appello, quia in ea nondum existit motus, sed tantum solicitation ad motum [...]; alia vero vis ordinaria est, cum motu actuali conjuncta, quam voco *vivam*."[14] (GM VI, 238)

Dead force is a theoretical primitive for Leibniz. It is an elementary "endeavor" or "tendency" to motion that is present both in bodies at rest and in bodies in motion. The former are conceived as objects that would move were some impediment to motion removed. Examples include a lever balanced by a counterweight, a stretched spring, or a body suspended from a height. In an object at rest, dead force is the force by which motion is initiated; in an object already moving, dead force accounts for changes in its state of motion and for the accumulation of the living force by which it acts on other bodies.

Leibniz illustrates the operation of dead force with the example of a tube rotating with a constant speed about a fixed center. At the end of the tube nearest the center, a ball is suspended. When released, the ball tends to move outward toward the other end of the tube. On Leibniz's analysis, prior to its release, the ball has dead force, in the form of a "conatus a centro recedendi," but this centrifugal conatus is "infinite parvum respectu impetus quem jam habet a

---

[14] "One force is elementary, which I also call *dead force*, since motion does not yet exist in it, but only a solicitation to motion […]. The other force is ordinary force, joined with actual motion, which I call *living force*" (AG 121).

rotatione" (GM VI, 238).[15] Upon its release, the ball acquires an outward motion, which increases through successive impressions of dead force until its centrifugal impetus is comparable to its rotational impetus.[16]

Figure 1

In this example, Leibniz makes several assumptions about the nature of dead force and its relation to motion:

- dead force is an infinitely small endeavor or tendency to motion;
- in the absence of impediments, the action of dead force produces an infinitely small change in a body's speed, and hence in its impetus;
- the accumulated effect of the action of dead force is a finite increase in a body's impetus.

So understood, the notion of dead force supports the analysis of motion in terms of infinitesimals. Assuming a continuous increase of speed, dead force is a body's power to pass from a resting speed v to a speed $v + dv$ in the interval dt. Because v itself is conceived as an infinitesimal change of distance, dead force can be construed as a second-order infinitesimal, which stands in the same relation of proportionality to speed and impetus as these first-order infinitesimals stand to motion and quantity of motion, respectively. Just as speed is conceived as

---

15 "It is obvious that, in the beginning, the conatus for receding from the center, […] is infinitely small in comparison with the impetus it already has from rotation" (AG 121).

16 Leibniz presents a fuller version of the argument in *Dynamica*, part II, props. 27-28 (GM VI, 451-52). The figure that follows in the text is reproduced from AG 121.

the infinitesimal distance traveled by a body in an instant, so dead force is an infinitesimal endeavor that effects an infinitely small change in a body's speed or impetus.[17]

Leibniz cites dead force as the cause not only of the generation of impetus but also of *vis viva*, or living force. The contribution that infinitesimal forces make to the production of these two types of dynamical property must be carefully distinguished. Solicitation, or dead force, impresses on a body an infinitesimal increment of speed *dv*, which can be greater or smaller depending upon the magnitude of the force. Impetus is generated through the addition of successive increments of speed; thus, assuming a constant force (as in free fall), impetus increases linearly with time. Descartes believed that a body's quantity of motion (or impetus) was a measure of its moving force, or its power to effect change in the state of another body through collision.[18] In his 1686 *Brevis demonstratio*, Leibniz showed that Descartes's measure of force cannot be correct, and that moving force is properly calculated as the product of a body's mass and the square of its speed.[19] It thus follows that in cases of uniform acceleration, where a body gains equal increments of speed in equal times, its living force increases in proportion to the square of time, or, equivalently, in proportion to the distance the body is moved. By the rules of Leibniz's calculus, we can infer that with each increment of speed *dv*, a body's living force increases by a factor of $2vdv$.[20] Therefore, while constant impressions of dead force over time produce a linear increase in the impetus of a moving body, they produce a geometrical increase

---

[17] "[J]ust as the numerical value [*aestimatio*] of a motion extending through time derives from an infinite number of impetuses, so, in turn, impetus itself (even though it is something momentary) arises from an infinite number of increments successively impressed upon a given mobile thing […]. From this it is obvious that the *nisus* is twofold, that is, elementary or infinitely small, which I also call *solicitation*, and that which is formed from the continuation or repetition of elementary nisus, that is, *impetus* itself" (GM VI, 238/AG 121). See also GM V, 325; GM VI, 451-52; GP II, 154; and GM VI, 151, translated in Bertoloni Meli, 1993, 131.

[18] Descartes, *Principia Philosophiae*, II, 40, 43 (AT VIII-1, 65-7).

[19] For analyses of the argument, see Brown, 1984; Garber, 1995, 310-13.

[20] An increment of speed *dv* produces a new force $F + dF$ that is proportional to $(v + dv)^2$, or $v^2 + 2vdv + dv^2$. Leibniz's rules for the calculus allow him to ignore the last infinitesimal product. Thus the newly acquired force is represented by the factor *2vdv*.

in its living force. Leibniz highlights this difference in a 1699 letter to De Volder, in which he draws on the symbolism of his calculus:

> Eodem modo etiam fit, ut gravi descendente, si fingatur ei quovis momento nova aequalisque dari celeritatis accessio infinite parva, vis mortuae simul et vivae aestimatio observetur, nempe ut celeritas quidem aequabiliter crescat secundum tempora, sed vis ipsa absoluta secundum spatia seu temporum quadrata, id est secundum effectus. Ut ita secundum analogiam Geometriae seu analysis nostrae solicitationes sint ut *dx*, celeritates ut *x*, vires ut *xx* seu ut $\int xdx$.[21] (GP II, 156)

Leibniz's conclusion that there is a significant difference in the mathematical representations of a body's quantity of motion and of its moving force is of fundamental importance for physics. Against Descartes, he stresses that the property of force involves two distinct components: a body's power at a moment to move itself or another body through a given *distance* and the *speed* at which it is able to do so. Hence, a body possesses greater living force to the extent that it is able to move a greater mass a greater distance, and to do so more quickly (GP II, 220). On Leibniz's account, impetus measures the speed that a body acquires through successive impressions of dead force, but it does not measure the contribution that dead force makes to a body's capacity to effect change in its own state of motion or in the state of motion of another body. That is measured by a body's living force. Unfortunately, beyond telling us that living force "ex infinitis vis mortuae impressionibus continuatis nata [arises from infinite continual impressions of dead force]" (AG, 122/GM VI, 238), and that dead force "vis vivae […] non nisi infinitesimalis pars est [is only an infinitesimal part of living force]" (AG, 255/GM VI, 104), Leibniz has little to say about how the two types of force are related.[22]

---

[21] "It happens in the same way also with a falling weight that a measure of both dead and living forces is obtained, if it is imagined that at any moment it receives a new and equal infinitely small increase in speed. Namely, the speed increases in equal amounts according to time, but the absolute force itself increases according to space or the square of the times, that is, in accordance with the effect. So, by analogy with geometry, or my analysis, solicitations are as *dx*, speeds are as *x*, and forces are as *xx* or $\int xdx$."

[22] The interpretation of their relation remains a matter of controversy. For a brief survey of the literature, see Bertoloni Meli, 1993, 89-90.

2. Infinitesimals as "Useful Fictions"

In the *Specimen dynamicum* Leibniz appeals to infinitesimal forces in explaining the initiation of motion, continuous increases or decreases of speed or impetus, and the accumulation of living force through motion. Yet despite the prominence he gives to these accounts, there is reason to doubt whether he intends them to be taken literally as descriptions of causal processes underlying physical change. In the *Specimen dynamicum* itself, he expresses reservations about interpreting infinitesimal quantities as real properties found in nature: "[…] non ideo velim haec Entia Mathematica reapse sic reperiri in natura, sed tantum ad accuratas aestimationes abstractione animi faciendas prodesse."[23] (GM VI, 238)

Although this passage may seem to suggest a special problem about the physical instantiation of infinitesimal quantities, Leibniz's early reflections on the "labyrinth of the continuum," and his subsequent efforts to absolve his differential calculus of any commitment to the reality of infinitesimals, demonstrate that his concerns are broader than this. In texts from the 1670s, Leibniz returns repeatedly to the paradoxes posed by the notion of infinitely small magnitudes, particularly infinitesimals of space and time. He ultimately concludes that the idea of a determinate magnitude that is smaller than any finite magnitude is an incoherent one. Thus, infinitesimals cannot be appealed to as basic elements from which a spatial or temporal continuum is composed.[24]

Following the publication of his calculus, Leibniz is forced to return to this topic. When critics attack the calculus because of its perceived reliance on infinitesimals, Leibniz responds by

---

[23] "I would not want to claim on these grounds that these mathematical entities are really found in nature, but I only wish to advance them for making careful calculations through mental abstraction." (AG 121) - See also GM IV, 91; GP II, 305; GP VI, 629/AG 230 (cited in n. 2).

[24] This commitment is in place by 1676. For discussion of the relevant arguments, see Arthur's Introduction to DLC, liv-lvii; Arthur, forthcoming; Levey, 1998, and Levey, 2003.

rejecting the charge. The calculus, including higher-order differentials, can be defended as a mathematical tool without Leibniz needing to commit himself to the reality of infinitesimals. The calculus itself is identified with a set of rules for differentiation and integration (or methods for finding tangents, quadratures, etc.), and the difference *dx* is understood not as a determinate, infinitely small magnitude, but as an indeterminate "differentiam duarum quantitatum communium ipsis quantitatibus incomparabilem."[25] (GM VI, 151). In appealing to an "incomparable" difference, Leibniz makes no assumption about the absence of a strict proportionality between the finite quantities and their difference. Rather, he claims that, though finite, the difference always can be taken to be sufficiently small that no error results from it. Thus, the validity of reasoning using the calculus does not presuppose the existence of actual infinitesimals. In place of infinitely small quantities, one can take the differences "tam parvas quam sufficere judicat, ut sint incomparabiles et errorem nullius momenti, imo dato minorem, producant."[26] (GM VI, 151).

With this interpretation of the calculus in place, Leibniz can argue that, while talk of infinitesimals may have some heuristic value, in the strictest sense, he regards them "pro mentis fictionibus [as fictions of the mind]" (GP II, 305). Unlike the continuum, they have no reality even as ideal entities, but are merely imaginary: entities that are feigned to exist. For this reason, there is no case to be made for infinitesimals either as real physical entities or as mathematical ones. In a 1702 letter to Varignon, Leibniz writes:

---

[25] "difference of two ordinary quantities, incomparable with the quantities themselves."

[26] "to be as small as one judges sufficient, so that they are incomparable and produce an error of no importance, indeed one smaller than any given." - This account is advanced publicly in the "lemmata" to his 1689 *Tentamen de Motuum Coelestium Causis*. See Bertoloni Meli, 1993, 130-31. Leibniz elaborates on it in his letter to Varignon of February 2, 1702 (GM IV, 91-2/L 543). For further discussion, see Jesseph, 1998.

> Pour dire le vray, je ne suis pas trop persuadé moy même, qu'il faut considerer nos infinis et infiniment petits autrement que comme des choses ideales ou comme des fictions bien fondées. Je croy qu'il n'y a point de creature au dessous de la quelle il n'y ait une infinite de creatures, cependant je ne crois point qu'il y en ait, ny même qu'il y en puisse avoir d'infiniment petites et c'est ce que je crois pouvoir demonstrer.[27] (GM IV, 110)

Leibniz categorically rejects the postulation of infinitely small parts of matter, and, we may assume, infinitely small quantities of force. The infinitely small does not exist as an element of the finite; rather, in the physical world there are only finite things, composed of smaller finite things, all the way down. He reiterates this view four years later to Des Bosses:

> Caeterum ut ab ideis Geometriae ad realia Physicae transeam, statuo materiam actu fractam esse in partes quavis data minores, seu nullam esse partem, quae non actu in alias sit subdivisa diversos motus exercentes. Id postulat natura materiae et motus et tota rerum compages, per physicas, mathematicas et metaphysicas rationes.[28] (GP II, 305)

If every part of matter is actually subdivided into smaller parts exercising different motions, then any quantity of force exerted by a part of matter will be a composite of the forces exerted by its parts. For Leibniz, however, this composition relation holds only among finite things. There is no ultimate account of the composition of finite things from infinitely small things, for infinitesimals of force or matter do not exist.

3. The Ideality of Finite Magnitudes

---

[27] "To speak the truth, I am not at all persuaded myself that it is necessary to consider the infinite and infinitely small other than as ideal things or as well-founded fictions. I believe that there is no created thing beneath which there is not an infinity of created things; however, I do not believe that any of them are, or even that any of them could be, infinitely small—and this I believe can be demonstrated."

[28] "To pass now from the ideas of geometry to the realities of physics, I hold that matter is actually fragmented into parts smaller than any given part; that is, there is no part of matter that is not actually subdivided into others exercising different motions. This is demonstrated by the nature of matter and motion and by the structure of the universe, for physical, mathematical, and metaphysical reasons." - See also his letter to Jacob Bernoulli of late August 1698 (GM III, 536).

There seems, then, to be good reason to discount Leibniz's speculations in the *Specimen dynamicum* and elsewhere about infinitesimal forces, and to focus instead on an understanding of force as a physical property that always possesses a finite magnitude. Can we in this way arrive at an understanding of force as "aliquid reale et absolutum"[29] (GM VI, 248)? In Leibniz's view, we cannot. Although the idea of a finite quantity is—in contrast to that of an infinitesimal quantity—a mathematically coherent one, it identifies a property that is, according to Leibniz, merely ideal. Hence, a representation of force as a finite quantity cannot be a representation of force insofar as it is "real and absolute."[30]

The argument for this conclusion hinges on a crucial premise: Given the role played by force in the explanation of physical change (change measured in terms of time and distance), the magnitude of any such force is represented as the value of a continuous function. As a body accelerates, it gains force continuously in proportion to the square of time; when a body loses force through collision with another body, it does so continuously, or "through degrees"—change never occurring through a leap. A conception of physical properties as finite quantities that vary continuously with respect to time and space is integral to the modern conception of physical theory. Leibniz saw this more clearly than most. However, he also insists that continuous quantities as such are merely *ideal*:

> Sed continua Quantitas est aliquid ideale, quod ad possibilia et actualia, qua possibilia, pertinet. Continuum nempe involvit partes indeterminatas, cum tamen in actualibus nihil sit indefinitum, quippe in quibus quaecunque divisio fieri potest, facta est.[31] (GP II, 282)

---

[29] "something real and absolute" (AG 131).

[30] Bracketed in this section is the problem of the duration of force. Even if a coherent conception of force as a finite quantity could be defended, there would still be the question of how such a force can be real, if only momentary. I return to this question in section 4.

[31] "Continuous quantity is something ideal, something that pertains to possibles and to actual things considered as possible. The continuum, of course, contains indeterminate parts. But in actual things nothing is indefinite, indeed, every division that can be made has been made in them" (AG 185).

The distinction between the actual or real (including the physically real) and the ideal is foundational to Leibniz's metaphysics; it is the escape route by which he extricates himself from the "labyrinth of the continuum." His decisive observation concerns the structure of a mathematical continuum. A genuine continuum cannot be conceived as resolvable into, or composed from, basic elements. Infinitesimals are a candidate for such elements, but Leibniz rejects them as incoherent. This is why the account of continuous change in terms of the action of infinitesimal forces has only a heuristic value. In the end, Leibniz concludes that it is an error to think of a continuum in mereological terms at all. There are no actual, determinate divisions within it, and there is no way to generate it through the summation of parts. Instead, it is the nature of a continuum that it is a whole in which the possibility of indefinitely many arbitrary divisions can be conceived—more divisions than are measurable by any countable sequence. This is a significant fact that Leibniz recognizes about the continuum. However, he is most struck by the metaphysical import of this fact. According to Leibniz, it is a mark of the real that it is composed of determinate parts, and that it is resolvable into a set of elements, or "true unities." Implicit in the latter claim is the assumption that whatever is real, is either a substance (an *unum per se*) or something whose existence can be explained in terms of the prior existence of substances (an *unum per aggregationem*). Since a continuum has no determinate parts and is not resolvable into elements, it cannot be real. Consequently, continua—including those of space and time—are only ideal, that is, the contents of ideas or concepts.[32]

Given this result, we can pose the following question about the property of force: Insofar as physical force is identified with a continuous quantity, or is represented as a function of

---

[32] See GP II, 268/AG 178; GP II, 282/AG 185; GP IV, 568/L 583.

continuous magnitudes, must it also be regarded as merely ideal? In one of his last letters to De Volder, Leibniz writes:

> […] in extensione Mathematica, […] nec prima Elementa, non magis quam inter numeros fractos minimus datur velut Elementum caeterorum. [Hinc Numerus, Hora, Linea, Motus seu gradus velocitatis, et alia hujusmodi Quanta idealia seu entia Mathematica revera non sunt aggregata ex partibus, cum plane indefinitum sit quo in illis modo quis partes assignari velit, quod vel ideo sic intelligi necesse est, cum nihil aliud significent quam illam ipsam meram possibilitatem partes quomodocunque assignandi.][33] (GP II, 276)

If continuous functions of time and distance, including a body's "degree of speed," are ideal quantities, then it would seem that the same should be said about force. An obvious objection to this inference is that, for Leibniz, force is not defined simply as a function of time and distance. His decisive objection to Descartes's physics is that force cannot be measured as the product of size and speed. The critical point here is usually taken to be Leibniz's claim that force is proportional not to speed, but to the square of speed (or velocity). No less significant, however, is his insistence that matter or mass cannot be identified with geometrical extension. If it could, then force would be exhaustively represented as a function of spatial and temporal variables, and we would have to conclude that force is merely ideal. Yet, for Leibniz, the "intimam corporum naturam [innermost nature of body]" (AG, 118/GM VI, 235) is not extension but *force*, and so there remains conceptual space for him to retain the idea that force itself is something real—and not merely, like space, time and degree of speed, ideal.

---

[33] "[…] in mathematical extension […] there are no basic elements, any more than a smallest number is found among the fractions, as the element of the rest. [Hence number, hour, line, motion or degree of speed, and other ideal quantities of this kind, that is, mathematical entities, are not in fact aggregated from parts, since the way in which someone may choose to assign parts in them is completely undetermined. Indeed, it is necessary that they be understood in this way, since they signify nothing other than the mere possibility of assigning parts in any way whatever.]" Leibniz indicates on his copy of the letter that the material enclosed in brackets was not included in the version sent to De Volder.

This clearly tracks the direction of Leibniz's thought. The problem, though, is how to conceive of, or to represent, force in a way that is consistent with our understanding of it as real. The charge, supported by Leibniz's analysis of the continuum, is that the resources of his science of dynamics do not allow us to do this. Any representation of a body's force as a value of a continuous function leaves us with a conception of force as ideal. For consider: for any body of fixed mass *m*, its force varies continuously as the square of its speed. Since *m* is constant, the only relevant factor in defining a body's acquisition of greater living force is its greater speed. Yet degree of speed is, according to Leibniz, an ideal quantity; hence, the increase in the body's force is represented in a way that we can only regard as ideal. Appeal to the magnitude of *m* is of no help, since it too is represented in the theory as the value of a continuous function, and in any case we have no access to *m* except via changes in a body's spatial and temporal parameters in response to the action of forces, themselves measured in terms of spatial and temporal parameters. In short, the science of dynamics offers us no way of representing a body's force—as a determinate finite quantity—that confirms its status for Leibniz as "something real and absolute."

Another way to the same conclusion is offered by Leibniz's assertion that any part of matter is actually subdivided into smaller parts *ad infinitum*. Here he wishes to emphasize that in matter as it exists, there is not merely, as there is in mathematical extension, "the possibility of division in any way whatever." Any part of matter is actually infinitely divided in some determinate way. For Leibniz, this marks matter as having a different ontological status than a spatial continuum, and we may suppose that this difference applies also to matter's "innermost nature": force. Just as any part of matter is subdivided into infinite actual parts, so the force of that matter is divided into infinitely many smaller discrete forces. For two reasons, however, this fact is of limited value in helping us to understand the reality of force. First, while matter (and force) is posited to have the structure of an infinite envelopment of discrete parts, the science of dynamics depends upon the assumption that physical change is *continuous* change. Hence the basic tools we have for conceiving of physical force represent it in a way that fails to support the claim made for its reality. Second, the actual division of matter to infinity satisfies only a necessary condition for its reality. In this way we are able to conceive of any part of matter as an aggregate of prior things. However, because matter is represented by us as inherently spatial, such a resolution into prior parts is always incomplete: any division produces parts which themselves are further divided. At no point in the spatial resolution of matter do we reach "true unities," or substances—the only entities in terms of which the reality of matter and its force can be demonstrated.[34]

4. The Reality of Force

For Leibniz, I suggest, there is no fully adequate representation of force as a mathematical quantity. In saying this, I do not deny that Leibniz believes that there are correct answers to the question of how force ought to be measured within physical theory. He is confident that his dynamics offers a correct measure of the moving force of a body and that Descartes's physics does not. What I do deny is that, for Leibniz, *any* mathematical formula expressible as a function of spatial and temporal variables is adequate to represent force as "something absolute and real." To represent force as an infinitesimal quantity, we have seen, is to represent it as something that is, strictly speaking, impossible. To represent it as a finite continuous quantity, is to represent it

[34] In Leibniz's late (post-1700) writings, the "true unities" that ground the reality of matter and force are mind-like monads. To Varignon, he writes in 1702: "The fact is that simple substances (that is, those which are not beings by aggregation) are truly indivisible, but they are immaterial and only principles of action" (GM IV, 110). I examine this position in detail in Rutherford, 2004, and Rutherford, 2008.

in a way that is mathematically coherent, but which involves a falsification of physical reality (it represents the real *as* ideal). These do not exhaust the possibilities of mathematical representation; we might suppose modeling the structure of matter/force through some form of discrete mathematics.[35] However, Leibniz does not envision this possibility and, in fact, makes it clear that he rejects any attempt to explicate force solely in mathematical terms. Force, instead, is something "metaphysical," which is not representable by the imagination, but can be grasped only by the intellect.[36] This is a point that many philosophers, particularly Cartesians, have failed to recognize, in Leibniz's view:

> Sed vulgo homines imaginationi satisfacere contenti rationes non curant, hinc tot monstra introducta contra veram philosophiam. Scilicet non nisi incompletas abstractasque adhibuere notiones sive mathematicas, quas cogitatio sustinet sed quas nudas non agnoscit natura, ut temporis, item spatii seu extensi pure mathematici, massae mere passivae, motus mathematice sumti etc.[37] (GP II, 249).

In order to understand force as *real*, we must set aside the imagination and mathematical modes of representation and rely instead on metaphysical concepts, known through the intellect. In the *Specimen dynamicum*, he writes:

> Hinc igitur, praeter pure mathematica et imaginationi subjecta, college quaedam metaphysica solaque mente perceptibilia esse admittenda, […] Id principium Formam, an 'εντελέχειαν, an

---

[35] This strategy is explored in Levey, 1998.

[36] In his *Lettre sur la Question si l'Essence du Corps Consiste dans l'Etendue*, published in the *Journal des Savans* in 1691, Leibniz writes: "there is in nature something other than what is purely geometrical, that is, extension and mere changes in it […]. It is necessary to join to it some higher or metaphysical notion, namely, that of substance, action and force" (GP IV, 465). See also GP VI, 507/AG 192: "the laws of force depend upon some marvelous principles of metaphysics or upon intelligible notions, and cannot be explained by material notions or the notions of mathematics alone or by those falling under the jurisdiction of the imagination."

[37] "People are generally content to satisfy their imaginations and do not worry about reasons; hence so many monstrosities are introduced to the injury of the true philosophy. It is obvious that they use only incomplete and abstract notions, or mathematical ones, which thought supports but which nature does not know in their bare form; such notions as that of time, also of space or of what is extended only mathematically, of merely passive mass, of motion considered mathematically, etc." (L 529).

> Vim appellemus, non refert, modo meminerimus per solam virium notionem intelligibiliter explicari.[38] (GM VI, 241-42).[39]

Leibniz's dynamics incorporates three metaphysical theses about the nature of force:

(1) the force that exists in matter is "quiddam prorsus reale" (GM VI, 247), and the "intimam corporum naturam" (GM VI, 235);[40]

(2) derivative force, or "quod in actione momentaneum est," is something "accidentale seu mutabile" (GP II, 270);[41]

(3) derivative force presupposes the existence of an active substance, or primitive active force, because "omne accidentale seu mutabile debet esse modificatio essentialis alicujus seu perpetui" (GP II, 270).[42]

We may see the third thesis as a way of reconciling the first two. As something accidental or changeable, derivative force is not a *per se* real being, or substance. The claim for its reality is justified, therefore, only if it is understood to exist as a modification of a prior substantial principle. In this way Leibniz moves the discussion of force squarely into the domain of metaphysics. The interpretative questions raised by this move are legion. Here I propose to

---

[38] "[W]e must admit something metaphysical, something perceptible by the mind alone over and above that which is purely mathematical and subject to the imagination […]. Whether we call this principle form or entelechy or force does not matter, as long as we remember that it can be intelligibly explained only through the notion of forces" (AG 125).

[39] On Leibniz's association of mathematics and metaphysics with different modes of cognition, see his 1702 letter to Queen Sophie Charlotte, *On What Is Independent of Sense and Matter* (GP VI, 500-2/AG 187-88). His claim that metaphysical concepts in general are grasped through reflective self-knowledge throws considerable light on his belief that the paradigm of a substance is an immaterial soul. See GP II, 270/AG 180-81; GP II, 276/AG 182; and Rutherford, 1995, 83-5.

[40] "something absolutely real" (AG 130); "innermost nature of body" (AG 118).

[41] "what is momentary in action" is "accidental or changeable" (AG 180).

[42] "everything accidental or changeable must be a modification of something essential or perpetual" (AG 180). Leibniz repeats this line of reasoning on many occasions: "we must consider derivative force (and action) as something modal, since it admits of change. But every mode consists of a certain modification of something that persists, that is, of something more absolute. […] Therefore, derivative and accidental or changeable force will be a certain modification of the primitive power [virtutis] that is essential and that endures in each and every corporeal substance" (GM VI 102-3/AG 254).

address only two of them. The first concerns the nature of the substance, or active principle, in terms of which Leibniz explains the reality of physical force; the second, the support this account offers for his claim that physical force is something "momentary."

To the extent that it intersects with his dynamics, Leibniz frames his theory of substance in a vocabulary inherited from Aristotle. The primary commitment of the theory is to a conception of substance as an original ground or principle of change.[43] The basis of a substance's fulfilling this function is its intrinsic power (*potentia*). However, against Aristotle, Leibniz insists that this power is not simply a potential or capacity for action, but a fully actual endeavor (*conatus*, *tendentia*). To mark its actuality, Leibniz labels this power, insofar as it is identified with a substance, *entelechy*, or *primitive active force*. And he contrasts this entelechy with *derivative* force, which includes all of the particular moments of "effort," by which a substance strives to attain new states.[44]

The idea of "effort" brings us back to the notion of dead force. We may recall that Leibniz defines dead force as an elementary endeavor or tendency to motion—a tendency evident in a compressed spring or a lever balanced by a counterweight. In elaborating the details of his dynamics, I stressed that dead force is explanatorily basic for Leibniz: it is the cause of any initiation of motion. Now, as we saw, in some of his writings, Leibniz attempts to subject the

---

[43] In the published essay *De ipsa natura* (1698), Leibniz poses the question "whether there is any *energeia* in created things." He responds by saying that he does not think "that it is in agreement with reason to deny all created, active force inherent in things"; and then continues: "Now let us examine a bit more directly […] that nature which Aristotle not inappropriately called the *principle of motion and rest*; though, having taken the phrase rather broadly, that philosopher seems to me to understand not only local motion or rest in a place, but *change* in general and stasis or persistence" (GP IV, 504-5/AG 156).

[44] "Active force, which one usually calls force in the absolute sense, should not be thought of as the simple and common potential [potentia] or receptivity to action of the schools. Rather, active force involves an effort [conatus] or striving [tendentia] toward action, so that, unless something else impedes it, action results. And properly speaking, entelechy, which is insufficiently understood by the schools, consists in this" (GM VI 101/AG 252). See also *Nouveaux Essais*, II.xxi.1(A VI, 6, 169/NE 169); *Nouveaux Essais*, II.xxii.11 (A VI, 6, 216/NE 216); *Theodicée*, §87 (GP VI, 149-50).

notion of dead force to the imagination, equating it with the force necessary to bring about an infinitely small change of speed. This move, I argued, cannot sustain a rigorous analysis, for the attempt to represent dead force as an infinitely small magnitude leads to the conclusion that dead force, strictly speaking, cannot exist. One response to this conclusion would be to say, so much the worse for dead force: like the infinitesimal itself, the notion may have some heuristic value, but it does not pick out any real entity in nature. In my view, this response is not supported by Leibniz's philosophy. Properly construed, as a moment of endeavor realized in an entelechy, dead force is indeed a cause of change. It is not the principle of change—that role is assigned to substance itself; but dead force is a determination or modification of that principle, and hence something (derivatively) real. This is a point we miss if we attempt to reduce a metaphysical concept to a mathematical function.

The challenge of linking Leibniz's theory of substance to the details of his dynamics manifests itself in a variety of ways. One problem arises from the fact that Leibniz explicitly cites the mind or soul as the paradigm of an entelechy: "la plus claire idée de la puissance active nous vient de l'esprit. Aussi n'est elle que dans les choses qui ont de l'analogie avec l'esprit, c'est-à-dire dans les Entelechies" (A VI, 6, 172).[45] Not all entelechies are minds for Leibniz, but all entelechies are mind-like—what he calls in his late writings "monads." Yet if primitive active force belongs exclusively to mind-like substances, whose derivative forces are strivings for new perceptual states, can this account be of any use to Leibniz in grounding the reality of physical forces:

[45] "the clearest idea of active power comes to us from the mind. So active power occurs only in things that are analogous to minds, that is, in entelechies" (*Nouveaux Essais*, II.xxi.4 (NE 172)). See also *Specimen dynamicum*, Part I: "primitive force (which is nothing but the first entelechy) corresponds to the *soul or substantial form*" (GM VI, 236/AG 119).

conatus, impetus and *vis viva*? Elsewhere, I have argued that Leibniz does have a story to tell here, though it is one that takes us deep into the arcana of his idealism.[46]

A more pressing problem concerns Leibniz's characterization of derivative force as something "momentary [*momentaneum*]." This term strongly suggests an attempt to predicate temporal properties of derivative force: in itself, derivative force exists only for a moment. Yet this seems a disastrous route for Leibniz to take. We have seen that he rejects any attempt to quantify duration or distance in terms of infinitely small magnitudes. So, what precisely could he mean in claiming that derivative forces are "momentary"?

There are two ways in which Leibniz might respond to this question, neither of which requires the ascription of a temporal duration to derivative forces. The first involves the (admittedly counterintuitive) idea that, as used in this context, the term *momentaneum* carries no temporal connotation at all. Instead, the term is to be understood in a sense related to the technical notion of a "moment": a tendency to produce motion about a point or axis.[47] To say that derivative force is *momentaneum*, then, would be to say merely that it is, or possesses, a tendency toward change, or the realization of a new state of a substance.

While this reading highlights an essential property of force—its nature as an inherent tendency—it arguably falls short of an adequate explanation of Leibniz's use of the term "momentary." One piece of evidence for this is that in his dynamical writings Leibniz establishes a mathematical relation between a body's momentary force or power and the temporal expression of that force—what he calls "action," defined as the product of power and time. In a 1713 letter to Hermann, he

---

[46] See Rutherford, 2004, 223-26. For Leibniz's defense of this strategy, see his letter to De Volder of June 30, 1704 (GP II, 270-71/L 537-38), and the discussion in Adams, 1994, 378-86.

[47] The source of this concept is Archimedes, whose law of the lever states that equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance. The moment is defined as the product of the applied force and the distance from the point of its application to the rotational axis. Thus, for equal forces, a longer arm will produce a greater moment, or tendency to motion.

writes: “At potentia mihi per tempus extenditur, quia ipsa per se, meo sensu, tempus non involvit, sed est momentaneum quiddam, quod quovis momento replicatur, seu ducitur in tempus. Et ita prodit actio data.”[48] (GM IV, 389).[49] At the very least, this passage affirms that momentary forces can be ascribed a temporal position—they exist before or after other momentary forces—and that in physics their summation over time is a measure of a body’s action. Nevertheless, it is notable that Leibniz is careful not to assign a temporal dimension to power itself (“ipsa per se […] tempus non involvit”). The tendency to change that is a body’s power does not last for any length of time, finite or infinitesimal. Hence, power is not to be construed as an element, or temporally minimal part, of action.

In describing derivative force as “something momentary,” I believe, Leibniz is best read as meaning not a tendency that exists *for* a moment, but a tendency that exists *at* a moment.[50] The temptation to think of derivative forces as existing for a moment stems from a basic confusion about their nature. We think of derivative forces in this way, because we imagine them as discrete existences that can be conceived independently of the primitive active force of substance. We picture them in the way that Leibniz describes dead force in the *Specimen dynamicum*, as separate moments of endeavor, whose effects are infinitesimal increments of speed or *vis viva*. This, however, is an error. From the perspective of metaphysics, derivative forces are nothing more than the primitive force of substance, *conceived as determined in some particular way*. Recall that, on Leibniz’s account, a substance’s power is not merely a potential

---

[48] “But power for me is extended through time, since, in my sense, in and of itself it does not involve time, but is something momentary, which is replicated at any moment or is prolonged in time. And in this way it produces a given action.”

[49] See also his letter to the same correspondent of September 9, 1712: “the notion of *power* is such that, multiplied by the time in which it is exercised, it produces action; that is, power is that whose temporal exercise is action, for power cannot be known except from action” (GM IV, 379).

[50] Though hardly decisive, Leibniz frames his view in this way in Part II of the *Specimen*: “since only force and the nisus arising from it exist at any moment [*quovis momento existat*] […]” (GM VI, 252/AG 135).

or capacity for acting; it is a fully determinate power, or entelechy, that is spontaneously exercised in action. Consequently, in designating a substance's derivative force, we are not referring to any entity over and above the substance itself; we are referring simply to some *way* or *mode* in which the substance exists—a mode in which it exhibits such-and-such tendency to change.[51]

The idea that derivative force is to be construed as a substance's tendency to change *at* a moment might suggest a general strategy for the interpretation of time-dependent physical properties. Contrary to Leibniz's assertion in the *Specimen dynamicum*, a body's speed should be conceived of not as an infinitesimal distance traveled in an instant, but as a tendency to change position *at* an instant. Likewise, a body's acceleration should be understood not as an infinitesimal increment of speed (or velocity), but as a tendency to change speed (or velocity) *at* an instant. Supporting this proposal is the fact that the modern interpretation of the calculus follows Leibniz in dispensing with infinitesimal differences, and replaces them with the notion of a derivative: a continuous function that delivers the value of a time-dependent variable at a moment.[52]

Such an approach would have some affinity with Leibniz's account of derivative force; however, it would not succeed in establishing kinematic properties as real for him. The reason for this, again, is his account of the ontology of space and time. If space and time are merely ideal, then

---

[51] "*Primitive powers* [Les *Puissances primitives*] constitute the substances themselves, and *derivative* powers, or faculties, if you like, are only *ways of being* [*façons d'estres*], which must be derived from substances" (*Nouveaux Essais* IV.iii.6 (A VI, 6, 379/NE 379)). "Derivative force is itself the present state when it tends toward or preinvolves a following state, as every present is pregnant with the future. But that which persists, insofar as it involves all cases, contains primitive force, so that primitive force is, as it were [velut], the law of the series, while derivative force is, as it were, a determination which designates some term in the series" (GP II, 262/L 533). Leibniz's terminology is not always consistent. In a later letter to the same correspondent (Burcher de Volder), he reserves the term "derivative force" for phenomenal physical forces, while uniting the substantial power and its tendencies under the heading of "primitive force": "I relegate derivative forces to the phenomena, but I think that it is obvious that primitive forces can be nothing but the internal tendencies [tendentias] of simple substances, tendencies by means of which they pass from perception to perception in accordance with a certain law of their nature" (GP II, 275/AG 181).

[52] For recent attempts to defend the reality of instantaneous velocity along these lines, see Smith, 2003; Lange, 2005. Lange, in particular, argues for the view that instantaneous velocity is best understood as a tendency..

so is any continuous function of spatial or temporal variables. Thus, while properties such as velocity and acceleration may be both mathematically and physically well defined, they do not meet the strictures of Leibniz's metaphysics; they do not pick out real properties of substance. According to Leibniz, derivative force *can* be understood in a way that is consistent with its status as a property of substance. Yet this is possible only insofar as derivative force is not represented mathematically in terms of spatial and temporal parameters, but is expressed in a properly metaphysical vocabulary.

In attempting to formulate such a vocabulary, Leibniz associates derivative force with the *states* of a substance. In studies from the 1680s, he defines "status [state]" as "rei praedicatum mutabile"[53] (A VI, 4, 633), and "mutatio [change]" as "complexum duorum statuum contradictoriorum sibi immediatorum"[54] (A VI. 4, 869). Any change thus marks a transition from one state of a substance to another. Leibniz provides few details about how states are to be individuated, but the clues he gives point to facts about their content as perceptual (or other mental) states.[55] Given this, we can surmise that there is a broad latitude in how we may pick out the states of a substance; in the case of a monad, we may do so by referring to any change in its perceptual contents. The crucial point is the link Leibniz establishes between a substance's states and its derivative force. Having picked out any pair of proximal, mutually incompatible states, we have thereby designated a moment of change, which is explained in terms of one state's

---

[53] "a changeable predicate of a thing."
[54] "an aggregate of two immediate, mutually contradictory states." Other variants of this definition: "Change is an aggregate of two contradictory states. But these states are necessarily understood to be immediate with respect to each other, since contradictories admit of no third thing" (A VI, 4, 556). "Change is an aggregate of two opposed states in one stretch of time, with no existing moment of change, as I demonstrated in a certain dialogue" (A VI, 4, 307). The dialogue to which Leibniz refers is the 1676 *Pacidius Philalethi*, which is one of the most important texts on this topic. For a detailed discussion, see Levey, 2003. See also A VI, 4, 563, 569, and 869, which expand the account to include a definition of temporal order, based on the causal connection of states.
[55] See, e.g., the 1679 *De affectibus*, where definitions of "mutatio" and "determinatio" appear in close proximity to definitions of mental states: "*Cogitatio* est status mentis qui conscientiae causa proxima est […]. *Sententia* est cogitatio ex qua sequitur conatus agendi ad externa" (A VI, 4, 1411).

inherent tendency to give way to another. In Leibniz's technical vocabulary, the prior state is a "determinatio," which is defined as "status ex quo quid sequitur nisi quid aliud impediat" [56] (A VI, 4, 1426); and its ability to fulfill this causal role is ascribed to its inherent force or endeavor. Drawing together the strands of his account, Leibniz writes: "Porro ipsa Transitio, seu variatio, […] nihil aliud est, quam complexus duorum statuum sibi oppositorum et immediatorum una cum vi seu transitus ratione, quae est ipsa qualitas"[57] (C, 9).[58]

With this account, according to which any state of a substance is endowed with an inherent tendency to change, we are again returned to the notion of dead force, now relocated to the internal dynamics of a substance. The primitive force of any substance is manifested in a continual succession of tendencies. This is evidenced both in a body's curvilinear motion and in a soul's series of perceptual states.[59] However, it is Leibniz's view, expressed with growing confidence in his later writings, that the internal dynamics of the soul offer a better vantage point

---

[56] "a state from which something follows unless something else impedes it."

[57] "Moreover, the change, or variation, itself […] is nothing but a complex of two states which are immediate and opposed to one another, together with a force or ground for the change, which itself is a quality."

[58] As I read Leibniz, there is no canonical description of a substance's states or the changes that occur in them. Any state description involves the attribution of a determinate predicate to the substance, but states may be designated in ways that are more or less finely grained. A coarse-grained description may involve the attribution of successive conscious states to a substance, but underlying these states there are changes in unconscious states, which would be registered by a more finely grained analysis. The doctrine of *petites perceptions*, moreover, suggests that there is no limit to how fine-grained the analysis might become. At each stage in the analysis, we would pick out salient differences that mark a change in a substance's states, but at no point would we arrive at a smallest difference that revealed the change to be essentially discontinuous (cf. *Nouveaux Essais*, Préface (A VI, 6, 56-7)). Does this mean, then, that the activity of a substance is to be understood as continuous, in the way that we think of physical processes (e.g., increases of speed or temperature) as continuous? There are problems with saying this, since as we have seen Leibniz is adamant that spatial and temporal continua are (merely) ideal. Nevertheless, he seems committed to saying that the activity of the substance itself (primitive active force) is expressed continuously, and that the derivative forces that indicate a substance's tendency to change at a moment are abstractions from that activity. Such forces designate the substance's activity in a way that is parasitic on the ascription of discrete states to it, when in point of fact such states do not exist (or exist only relative to a certain mode of conceiving). Arthur reaches a similar conclusion about "the real continuity of substantial activity" in his Introduction to DLC, lxxxvii.

[59] See Leibniz's reply to the second edition of Bayle's *Dictionary*: "The state of the soul, like that of the atom, is a state of change, a tendency. The atom tends to change its place, the soul to change its thoughts; each changes by itself in the simplest and most uniform way which its state permits" (GP IV, 562/L 579). Similar ideas are expressed at A VI, 4, 1426; GP II, 172/AG 172-73.

from which to grasp the reality of force than its external manifestations in bodily motion.[60] From this perspective we are able to recognize that derivative force cannot be separated, or even sharply distinguished, from the primitive force of substance. Primitive force is neither resolvable into, nor composed from, its successive tendencies. Rather, the tendencies are what it is to be that endeavor at some moment, defined in terms of proximal pairs of mutually exclusive states. Getting clear on the exact relationship between primitive and derivative force remains a task for Leibniz scholarship. I do not claim to have fully resolved that problem, but only to have drawn attention to the inadequate (from the point of view of metaphysics) conceptions of force and tendency that inform Leibniz's science of dynamics, particularly when those conceptions are tethered to the mathematics of the infinitesimal calculus.[61]

Bibliographical References

Adams, Robert M., 1994, *Leibniz: Determinist, Theist, Idealist,* New York: Oxford University Press.

Aiton, Eric J., 1984, "The Mathematical Basis of Leibniz's Theory of Planetary Motion", in: Albert Heinekamp (Ed.), *Leibniz's Dynamica* (*Studia Leibnititia*, Sonderheft 13), Stuttgart: Franz Steiner, 209-25.

Arthur, Richard T. W., forthcoming, "From Actuals to Fictions: Four Phases in Leibniz's Early Thought on Infinitesimals", Forthcoming in *Studia Leibnitiana*, Sonderheft volume, Mark Kulstad and Moegens Laerke (Eds.).

Bell, J. L., 1998, *A Primer of Infinitesimal Analysis*, Cambridge: Cambridge University Press.

[60] *Nouveaux Essais*, II.xxi.72 (A VI, 6, 210-11). From the perspective of the soul, a substance's endeavor is teleologically structured: it is a striving *for* the apparent good. See Nouveaux Essais II.xxi.5: "volition is the effort or endeavor [conatus] to move towards what one finds good and away from what one finds bad, the endeavor arising immediately out of one's awareness of those things […]. There are other efforts, arising from insensible perceptions, which we are not aware of; I prefer to call these 'appetitions' rather than volitions" (A VI, 6, 172-73/NE 172-73). I discuss this point in greater detail in Rutherford, 2005.

[61] I am grateful to Dan Garber, John Whipple, and the participants in my 2006 Leibniz seminar at UCSD for discussion of some of the issues examined in this essay. Thanks are owed also to Ursula Goldenbaum and Doug Jesseph for helpful comments on the penultimate draft.

Bertoloni Meli, Domenico, 1993, *Equivalence and Priority: Newton Versus Leibniz*, Oxford: Clarendon Press.

Breger, Herbert, 1984, "Elastizität als Strukturprinzip der Materie bei Leibniz," in: Albert Heinekamp (Ed.), *Leibniz's Dynamica* (*Studia Leibnitiana*, Sonderheft 13), Stuttgart: Franz Steiner, 112-21.

Brown, Gregory, 1984, "'*Quod ostendendum susceperamus*': What Did Leibniz Undertake to Show in the *Brevis Demonstratio*?" in: Albert Heinekamp (Ed.), *Leibniz's Dynamica* (*Studia Leibnitiana*, Sonderheft 13), Stuttgart: Franz Steiner, 122-37.

Cajori, Florian, 1928-29, *A History of Mathematical Notations*, 2 vols, Chicago: Open Court.

Garber, Daniel, 1995, "Leibniz: Physics and Philosophy", in: Nicholas Jolley (Ed.), *The Cambridge Companion to Leibniz*, Cambridge: Cambridge University Press, 270-352.

Jesseph, Douglas M., 1998, "Leibniz on the Foundations of the Calculus: The Question of the Reality of Infinitesimal Magnitudes", in: *Perspectives on Science* 6, 6-40.

Lange, Marc, 2005, "How Can Instantaneous Velocity Fulfill Its Causal Role?", in: *Philosophical Review* 114, 433-68.

Levey, Samuel, 1998, "Leibniz on Mathematics and the Actually Infinite Division of Matter", in: *Philosophical Review* 107, 49-96.

Levey, Samuel, 2003, "The Interval of Motion in Leibniz's *Pacidius Philalethi*", in: *Noûs* 37, 371-416.

Rutherford, Donald, 1995, *Leibniz and the Rational Order of Nature*, New York: Cambridge University Press.

Rutherford, Donald, 2004, "Idealism Declined: Leibniz and Christian Wolff", in: Paul Lodge, (Ed.), *Leibniz and His Correspondents*, Cambridge: Cambridge University Press, 214-37.

Rutherford, Donald., 2005, "Leibniz on Spontaneity", in: Donald Rutherford and J.A. Cover, (Ed.), *Leibniz: Nature and Freedom*, New York: Oxford University Press, 156-80.

Rutherford, Donald, 2008, "Leibniz as Idealist", in: *Oxford Studies in Early Modern Philosophy* 4, forthcoming.

Smith, Sheldon, 2003, "Are Instantaneous Velocities Real and Really Instantaneous?: An Argument for the Affirmative", in: *Studies in the History and Philosophy of Modern Physics*, 34, 261-80.